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Universal Gauss-Thakur sums and L-series [*†]

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Abstract. In this paper we study the behavior of the function ω of Anderson-Thakur (introduced in [1]) evaluated at the elements of the algebraic closure $\mathbb{F}_q^{\mathrm{alg}}$ of the finite field with q elements $\mathbb{F}_q$. Indeed, this function has quite a remarkable connection with explicit class field theory for the field $K = \mathbb{F}_q(\theta)$. We will see that these values, together with the values at $\mathbb{F}_q^{\mathrm{alg}}$ of its divided derivatives, generate over $\mathbb{F}_q^{\mathrm{alg}}$ the maximal abelian extension of K which is tamely ramified at infinity. We will also see that ω is, in a way that we will explain in detail, an *universal Gauss-Thakur* sum. We will then use these results to show the existence of functional relations for a class of L-series introduced by the second author in [14]. Our results will be finally applied to obtain a new class of congruences for Bernoulli-Carlitz fractions, and an analytic conjecture is stated, implying an interesting behavior of such fractions modulo prime ideals of $A = \mathbb{F}_q[\theta]$.

Contents

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1 Introduction, results

The present paper is divided in two parts (Section 2 for the first part and Sections 3 and 4 for the second), both motivated by the interesting behavior that the function ω of *Anderson and Thakur* ([1]) exhibits at the roots of unity, and the consequence that this behavior has on analytic properties of certain L-series introduced in [14].

We will first be concerned with the values of the function ω at the roots of unity and we will prove, among several results, Theorem 1, which provides, we hope, an alternative approach to *explicit class field theory.* We will also prove, in Theorem 3, that ω is, in a certain sense that will be made more precise later, an *universal Gauss-Thakur sum.*

Theorem 3 will be used in in the second part, where we will consider a class of L-series that was recently introduced by the second author in [14] and we will study their behavior at the roots of unity. We will prove, in Theorem 4, functional identities in the same vein as in loc. cit., but in a much more general multivariable setting. Among others, some applications to *Bernoulli-Carlitz* numbers are given, in Theorem 5.

Here is, more specifically, the content of the present paper. Our purpose in Section 2, the first part of this paper, is to focus on explicit class field theory for the field K. The classical Kronecker-Weber theorem states that the maximal abelian extension $\mathbb{Q}^{\mathrm{ab}}$ of the field of rational numbers $\mathbb{Q}$ in the field of complex numbers $\mathbb{C}$ is generated by the values of the exponential function

$$e^z = \sum_{n\geq 0} \frac{z^n}{n!}$$

at the elements $\sqrt{-1}\pi\rho$, $\rho \in \mathbb{Q}$, or, in other words, by the complex roots of the polynomials

$$X^n - 1, \quad n \geq 1.$$

The prominency of an analytic function in an algebraic problem is the essence of Kronecker's *Jugendtraum* (it later became the *twelfth Hilbert's problem*) and was confirmed in other situations by other authors, namely by Hayes in 1974, which in [11] analytically expressed a minimal set of generators of the maximal abelian extension of the field $K = \mathbb{F}_q(\theta)$ tamely ramified at infinity by means of the torsion values of *Carlitz's exponential function*, and further constructed the maximal abelian extension of K again relating it to the torsion of Carlitz's module.

[1] Introduced in [1].

We denote by v_∞ the θ^{-1}-adic valuation normalized, in all the following, by setting $v_\infty(\theta) = -1$. Let K_∞ be the completion of K for v_∞, and let us consider the completion $\mathbb{C}_\infty$ of an algebraic closure of K_∞ for the unique extension of this valuation, in which we embed an algebraic closure of K. Carlitz's exponential function is the surjective, $\mathbb{F}_q$-linear, rigid analytic entire function

$$\exp : \mathbb{C}_\infty \to \mathbb{C}_\infty$$

defined by

$$\exp(z) = \sum_{n\geq 0} \frac{z^{q^n}}{d_n},$$

where

$$d_0 = 1, \quad d_n = (\theta^{q^n} - \theta)(\theta^{q^n} - \theta^q)\cdots(\theta^{q^n} - \theta^{q^{n-1}}), \quad n > 0.$$

The kernel of this function turns out to be generated by a *period* $\widetilde{\pi}$, unique up to multiplication by an element of $\mathbb{F}_q^\times$, that can be computed by using the following product expansion

$$\widetilde{\pi} := \theta(-\theta)^{\frac{1}{q-1}} \prod_{i=1}^{\infty} (1 - \theta^{1-q^i})^{-1} \in (-\theta)^{\frac{1}{q-1}} K_\infty, \tag{1}$$

once a $(q-1)$-th root of $-\theta$ is chosen.

Anderson-Thakur function. This function, introduced in [1, Proof of Lemma 2.5.4 p. 177], is defined by the infinite product

$$\omega(t) = (-\theta)^{\frac{1}{q-1}} \prod_{i\geq 0} \left(1 - \frac{t}{\theta^{q^i}}\right)^{-1} \tag{2}$$

(it is customary to make the same choice as in (1) for the $(q-1)$-th root), converges for $t \in \mathbb{C}_\infty$ such that $|t| \leq 1$ (where $|\cdot|$ is an absolute value associated to v_∞) and can be extended to a non-vanishing rigid analytic function over

$$\mathbb{C}_\infty \setminus \{\theta^{q^k}; k \geq 0\}$$

with simple poles at θ^{q^k}, $k \geq 0$. In [16], many analogies with *Euler's gamma function* are tracked. For instance, variants of the *translation formula*, *Gauss multiplication formulas* and *reflection formulas* for the gamma function hold for ω. We are going to study yet another property of ω.

For $\zeta \in \mathbb{F}_q^{\mathrm{alg}}$, the product $\omega(\zeta)$ in (2) converges to an algebraic element of $\mathbb{C}_\infty$. More generally, consider the $\mathbb{C}_\infty$-linear *divided derivatives*

$$\mathcal{D}_n : \mathbb{C}_\infty[[t]] \to \mathbb{C}_\infty[[t]], \quad n \geq 0$$

defined by setting

$$\mathcal{D}_n t^m = \binom{m}{n} t^{n-m}.$$

Then, for all $n \geq 0$, the formal series $\mathcal{D}_n\omega$ converges for all $t \in \mathbb{C}_\infty$ such that $|t| \leq 1$. For $\zeta \in \mathbb{F}_q^{\mathrm{alg}}$, the series $(\mathcal{D}_n\omega)(\zeta)$ converges in fact to an element of K^{alg} (these properties will be apparent in the paper). Furthermore, let E^∞ be the smallest subfield of $\mathbb{C}_\infty$ containing the algebraic closure $\mathbb{F}_q^{\mathrm{alg}}$ of $\mathbb{F}_q$ in $\mathbb{C}_\infty$ and the values

$$\exp(\widetilde{\pi}\rho), \quad \rho \in K.$$

The first result of this paper is the following.

Theorem 1 *The field E^∞ is also generated over $\mathbb{F}_q^{alg}$ by the values $(\mathcal{D}_n\omega)(\zeta)$ for all $\zeta \in \mathbb{F}_q^{alg}$ and $n \geq 0$.*

According to Hayes [11], E^∞ is equal to the maximal abelian extension of K tamely ramified at ∞ in $\mathbb{C}_\infty$ ([2]), which obviously yields the next Corollary.

Corollary 2 *The higher derivatives of the function ω evaluated at the elements of $\mathbb{F}_q^{alg}$ generate, over $\mathbb{F}_q^{alg}$, the maximal extension of K which is abelian and tamely ramified at the infinity place.*

In the proof of Theorem 1, new functions generalizing the function ω are introduced. These are the functions $\omega_{\mathfrak{a},j}$ of Subsection 2.2. They generalize the function $\omega = \omega_{\theta,0}$ in the sense of Proposition 15 and should be considered of same relevance as ω itself, being associated to the kernel of $\phi_{\mathfrak{a}}$ (the image of a monic polynomial $\mathfrak{a} \in A$ by Carlitz's module) in the same way as ω is associated to the kernel of ϕ_θ. These features will be discussed in detail in Subsection 2.2.

The second result of Section 2, closely related to Theorem 1, draws a portrait of Anderson-Thakur's function itself, as an *universal Gauss-Thakur sum*. This analogue of Gauss sums, in K^{ab}, was introduced by Thakur in [21]. Thakur established several analogues of classical results about Gauss sums such as Stickelberger factorization theorem and Gross-Koblitz formulas and other analogues of classical results (see for example [21, 22, 23]). We refer to Subsection 2.1.1 for the background on Gauss-Thakur sums. We are going to describe a direct connection between Gauss-Thakur sums and the function ω.

Let $\mathfrak{p}$ be an irreducible monic polynomial of A of degree d, let $\Delta_{\mathfrak{p}}$ be the Galois group of the $\mathfrak{p}$-cyclotomic function field extension $K(\lambda_{\mathfrak{p}})$ of K, where $\lambda_{\mathfrak{p}}$ is a non zero $\mathfrak{p}$-torsion element of K^{alg}. Gauss-Thakur sums can be associated to the elements of the dual character group $\widehat{\Delta}_{\mathfrak{p}}$ via the Artin symbol (see [9, Sections 7.5.5 and 9.8]). If χ is in $\widehat{\Delta}_{\mathfrak{p}}$, we denote by $g(\chi)$ the associated Gauss-Thakur sum. In particular, we have the element $\vartheta_{\mathfrak{p}} \in \widehat{\Delta}_{\mathfrak{p}}$ obtained by reduction of the *Teichmüller character* [9, Definition 8.11.2], uniquely determined by a choice of a root ζ of $\mathfrak{p}$, and the Gauss-Thakur sums $g(\vartheta_{\mathfrak{p}}^{q^j})$ associated to its q^j-th powers, with $j = 0, \ldots, d-1$, which can be considered as the building blocks of the Gauss-Thakur sums $g(\chi)$ for general $\chi \in \widehat{\Delta}_{\mathfrak{p}}$.

Theorem 3 *Let $\mathfrak{p}$ be a prime element of A of degree d and ζ a root of $\mathfrak{p}$ as above. We have:*

$$g(\vartheta_{\mathfrak{p}}^{q^j}) = \mathfrak{p}'(\zeta)^{-q^j}\omega(\zeta^{q^j}), \quad j = 0, \ldots, d-1.$$

In this theorem, $\mathfrak{p}'$ denotes the derivative of $\mathfrak{p}$ with respect to θ. We anticipate that Theorem 3 will play an important role in the proof of the next Theorem 4. Also, The Theorems 1 and 3 are closely related. We will see, by Corollary 29 later in this paper, that the field generated over $\mathbb{F}_q^{\text{alg}}(\theta)$ by the various Gauss-Thakur sums $g(\vartheta_{\mathfrak{p}})$, is also equal to the field generated over $\mathbb{F}_q^{\text{alg}}$ by the elements $\lambda \in K$ which are a-torsion for $a \in A$ squarefree. But the proof of Theorem 1 that we furnish, founded on an analytic formula (Proposition 19), also tells us that the last field is generated, over $\mathbb{F}_q^{\text{alg}}$, by the elements $\omega(\zeta)$, $\zeta \in \mathbb{F}_q^{\text{alg}}$ (see Proposition 23).

In Section 3 we keep studying the values of ω at the elements of $\mathbb{F}_q^{\text{alg}}$, but we change our point of view by focusing now on certain L-series introduced in [14]. Let t be a variable in $\mathbb{C}_\infty$ and let us consider the ring homomorphism

$$\chi_t : A \to \mathbb{F}_q[t]$$

[2]That is, the maximal abelian extension of K whose perfection is contained in the subfield of Newton-Puiseux series $\cup_{n\geq 1}\mathbb{F}_q^{\text{alg}}((\theta^{-1/n}))$.

defined by the formal replacement of θ by t. In other words, χ_t may be viewed as the unique ring homomorphism from A to the ring of rigid analytic functions $\mathbb{C}_\infty \to \mathbb{C}_\infty$ such that $\chi_t(\theta) = t$. More generally, we shall consider s independent variables $t_1, \ldots, t_s$ and consider the ring homomorphisms

$$\chi_{t_i} : A \to \mathbb{F}_q[t_1, \ldots, t_s], \quad i = 1, \ldots, s$$

defined respectively by $\chi_{t_i}(\theta) = t_i$. To simplify our notations, we will write $\chi_\xi(a)$ or $a(\xi)$ for the evaluation at $t = \xi$ of the polynomial function $\chi_t(a)$ at a given element $\xi \in \mathbb{C}_\infty$. Let α be a positive integer and let $\beta_1, \ldots \beta_s$ be non-negative integers. The following formal series was introduced in [14]:

$$L(\chi_{t_1}^{\beta_1} \cdots \chi_{t_s}^{\beta_s}, \alpha) = \sum_{d \geq 0} \sum_{a \in A^+(d)} \chi_{t_1}(a)^{\beta_1} \cdots \chi_{t_s}(a)^{\beta_s} a^{-\alpha} \in K_\infty[[t_1, \ldots, t_s]]. \tag{3}$$

Here and in all the following, $A^+(d)$ denotes the set of monic polynomials of A of degree d. It is easy to see that this series is well defined. As claimed in [14, Remark 7], this series converges for all $(t_1, \ldots, t_s) \in \mathbb{C}_\infty^s$ to a rigid analytic entire function of s variables $t_1, \ldots, t_s$; see Proposition 32.

For the next result, we need further notation. For k a non-negative integer, we consider the q-ary expansion $k = k_0 + k_1 q + \cdots + k_s q^s$, where $k_0, k_1, \ldots, k_s$ are integers in the set $\{0, \ldots, q-1\}$. We then denote by $\ell_q(k)$ the integer $k_0 + k_1 + \cdots + k_s$. The residue of $\omega(t)$ at $t = \theta$ is $-\widetilde{\pi}$:

$$\widetilde{\pi} = -\lim_{t \to \theta} (t - \theta)\omega(t).$$

In [14, Theorem 1], it is proved that

$$L(\chi_t, 1) = \frac{\widetilde{\pi}}{(\theta - t)\omega(t)}.$$

Taking into account the functional equation

$$\omega(t)^q = (t^q - \theta)\omega(t^q)$$

apparent in (2), this implies that, for $m \geq 0$ integer,

$$V_{q^m,1}(t) := \widetilde{\pi}^{-q^m} L(\chi_t, q^m)\omega(t) = \frac{1}{(\theta^{q^m} - t)(\theta^{q^{m-1}} - t) \cdots (\theta - t)}.$$

This result provides an awaited connection between the function ω of Anderson and Thakur and the "positive even" values of the *Goss zeta function* (or *Carlitz zeta values*)

$$\zeta(n) = \frac{BC_n \widetilde{\pi}^n}{\Pi(n)}, \quad n > 0, \quad n \equiv 0 \pmod{q-1}$$

where BC_n and $\Pi(n)$ denote respectively the n-th Bernoulli-Carlitz fraction and Carlitz's factorial of n, see Goss' book [9, Section 9.1]. Indeed, evaluating at $t = \theta$, we get

$$L(\chi_\theta, q^m) = \zeta(q^m - 1), \quad m \geq 1.$$

More generally, it is proved in [14, Theorem 2] that, if $\alpha \equiv 1 \pmod{q-1}$ and $\alpha \geq 1$, then

$$\lambda_\alpha = \widetilde{\pi}^{-\alpha} L(\chi_t, \alpha)\omega(t)$$

is a rational function in $\mathbb{F}_q(\theta,t)$. In [14], it is suggested that this result could be a source of information in the study of the arithmetic properties of the Bernoulli-Carlitz fractions. However, the methods of loc. cit. (based on *deformations of vectorial modular forms* and Galois descent) are only partially explicit.

More recently, Perkins [17] investigated the properties of certain *special polynomials* associated to variants of the functions $L(\chi_t^\beta,\alpha)$ with $\alpha\leq 0$ which turn out to be polynomial. He notably studied the growth of their degrees. Moreover, by using *Wagner's interpolation theory* for the map χ_t, Perkins [18] generalized some unpublished formulas of the second author and obtained explicit formulas for the series

$$L(\chi_{t_1}\cdots\chi_{t_s},\alpha),\quad \alpha>0,\quad 0\leq s\leq q,\quad \alpha\equiv s \pmod{q-1}.$$

We quote here a particular case of Perkins' formulas for the functions $L(\chi_t,\alpha)$ with $\alpha\equiv 1 \pmod{q-1}$:

$$L(\chi_t,\alpha)=\sum_{j=0}^{\mu} d_j^{-1}(t-\theta)(t-\theta^q)\cdots(t-\theta^{q^{j-1}})\zeta(\alpha-q^j)L(\chi_t,q^j), \tag{4}$$

where μ is the biggest integer such that $q^\mu\leq\alpha$. It seems difficult to overcome the threshold $s\leq q$ giving at once expressions for $L(\chi_{t_1}\cdots\chi_{t_s},\alpha)$ with the effectiveness of Perkins' results.

In the next Theorem, we extend the previous results beyond the mentioned threshold, providing at once new quantitative information.

Theorem 4 *Let α,s be positive integers, such that $\alpha\equiv s \pmod{q-1}$. Let δ be the smallest positive integer such that, simultaneously, $q^\delta-\alpha\geq 0$ and $s+\ell_q(q^\delta-\alpha)\geq 2$. The formal series:*

$$V_{\alpha,s}(t_1,\ldots,t_s)=\widetilde{\pi}^{-\alpha}L(\chi_{t_1}\cdots\chi_{t_s},\alpha)\omega(t_1)\cdots\omega(t_s)\prod_{i=1}^{s}\prod_{j=0}^{\delta-1}\left(1-\frac{t_i}{\theta^{q^j}}\right)\in K_\infty[[t_1,\ldots,t_s]] \tag{5}$$

is in fact a symmetric polynomial of $K[t_1,\ldots,t_s]$ of total degree $\delta(\alpha,s)$ such that

$$\delta(\alpha,s)\leq s\left(\frac{s+\ell_q(q^\delta-\alpha)}{q-1}\right)-s.$$

This statement holds if $\alpha=q^m$ and $s\geq 2$ (so that $\delta=m$) assuming that empty products are equal to one by convention. In this case, since $s\equiv\alpha \pmod{q-1}$, we have $s+\ell_q(q^\delta-\alpha)\equiv 0 \pmod{q-1}$ so that in fact, $s\geq\max\{2,q-1\}$. The reader may have noticed that the choice $\alpha=q^m$ and $s=1$ is not allowed in Theorem 4. However, as mentioned above, the computation of $V_{q^m,1}$ is completely settled in [14]. This discrimination of the case $\alpha=q^m, s=1$ should not be surprising neither; similarly, the Goss zeta function associated to A has value 1 at zero, but vanishes at all negative integers divisible by $q-1$.

In Section 4, we will be more specifically concerned with Bernoulli-Carlitz numbers. A careful investigation of the polynomials $V_{1,s}$ and an application of the digit principle to the function ω will allow us to show that, for $s\geq 2$ congruent to one modulo $q-1$,

$$\mathbb{B}_s=\Pi(s)^{-1}V_{1,s}(\theta,\ldots,\theta)$$

is a polynomial in $\mathbb{F}_q[\theta]$ (Proposition 44) ([3]). We shall then show the next Theorem, which highlights the interest of these polynomials in θ.

[3]Note that $\mathbb{B}_1$ is not well defined

Theorem 5 *Let $s \geq 2$, $s \equiv 1 \pmod{q-1}$. Let us consider the expansion $s = \sum_{i=0}^{r} s_i q^i$ of s in base q. Let d be an integer such that $q^d > s$ and let $\mathfrak{p}$ be a prime of degree d. Then:*

$$\mathbb{B}_s \equiv \frac{(-1)^s BC_{q^d-s} \prod_{i=0}^{r} l_{d-i-1}^{s_i q^i}}{\Pi(q^d - s)} \pmod{\mathfrak{p}}.$$

In this result, l_d denotes the polynomial $(-1)^d \prod_{i=1}^{d}(\theta^{q^i} - \theta)$; we observe that the latter polynomial is invertible modulo $\mathfrak{p}$ just as $\Pi(q^d - s)$. The non-vanishing of $\mathbb{B}_s$ for fixed s signifies the existence of an explicit constant $c > 0$, depending on s and q, such that for all $d \geq c$,

$$BC_{q^d-s} \not\equiv 0 \pmod{\mathfrak{p}}, \qquad \text{for all } \mathfrak{p} \text{ such that } \deg \mathfrak{p} = d. \tag{6}$$

However, the non-vanishing of $\mathbb{B}_s$ is also equivalent to the fact that the function

$$L(\chi_{t_1} \cdots \chi_{t_s}, 1) \prod_{i=1}^{s} (t_i - \theta)^{-1},$$

entire of s variables (as we will see), is a unit when identified to an element of $\mathbb{C}_\infty[[t_1 - \theta, \ldots, t_s - \theta]]$; we presently do not know how to prove this property for all s. Therefore, the property (6) is linked with the following conjecture of nature analogue of classical results on the simplicity of the zeroes of Goss zeta functions and L-series, which should be, we believe, true.

Conjecture 6 *Let $s \geq 2$ be congruent to one modulo $q - 1$. Then, locally at $t_1 = \cdots = t_s = \theta$, the divisor of the zeroes of the function $L(\chi_{t_1} \cdots \chi_{t_s}, 1)$ is equal to the set of zeroes of the polynomial $\prod_i (t_i - \theta)$.*

Numerical computations on Bernoulli-Carlitz fractions made by Taelman provide some evidence to support this hypothesis. The Conjecture follows from Perkins results [18] in the case $s \leq q$ and $\alpha = s$. The conjecture is also verified if $\ell_q(s) = q$ and $\alpha = 1$, thanks to our Corollary 46.

We end this introduction with a general remark about our methods. One of the features of this paper is the analysis of problems involving several variables (especially in Section 3, but not only). Far from being a technical complication, this is crucial in our approach and may be difficult to avoid. Many corollaries we obtain by specialization in results in several variables that we obtain seem difficult to prove directly.

2 Algebraic values of the function of Anderson and Thakur

In this Section, we are going to pursue our investigation on the values of ω at the roots of unity and we will prove Theorems 1 and 3. Before going on, we collect an amount of known facts and necessary notations as well as well known definitions used all along this paper, for convenience of the reader.

2.1 Preliminaries

In this paper, we call *Carlitz's module* the unique $\mathbb{F}_q$-algebra homomorphism

$$\phi : A \to \mathbf{End}_{\mathbb{F}_q-\text{lin.}}(\mathbb{G}_a(\mathbb{C}_\infty))$$

determined by

$$\phi_\theta = \theta + \tau,$$

with τ the endomorphism such that $\tau(c) = c^q$ for all $c \in \mathbb{C}_\infty$. We also recall that the $\mathbb{F}_q$-algebra

$$\mathbf{End}_{\mathbb{F}_q-\text{lin.}}(\mathbb{G}_a(\mathbb{C}_\infty))$$

can be identified with the skew polynomial ring $\mathbb{C}_\infty[\tau]$ whose elements are finite sums $\sum_{i\geq 0} c_i\tau^i$ with the c_i's in $\mathbb{C}_\infty$, submitted to the usual product rule. We shall write ϕ_a for the evaluation of ϕ at $a \in A$.

For $a \in A \setminus \{0\}$, we set

$$\lambda_a = \exp\left(\frac{\widetilde{\pi}}{a}\right).$$

This is a generator of the kernel

$$\Lambda_a = \{\phi_b(\lambda_a); b \in A\}$$

of ϕ_a in $\mathbb{C}_\infty$, an A-module isomorphic to A/aA.

In all the following, a monic irreducible element in A will be called a *prime*. Let $\mathfrak{p}$ be a prime of $A = \mathbb{F}_q[\theta]$ of degree d. We denote by $K_\mathfrak{p} = K(\Lambda_\mathfrak{p}) = K(\lambda_\mathfrak{p})$ the $\mathfrak{p}$-th cyclotomic function field extension of K in $\mathbb{C}_\infty$. We refer the reader to [19, Chapter 12] for the basic properties of cyclotomic function fields. We recall here that the integral closure $\mathcal{O}_{K_\mathfrak{p}}$ of A in $K_\mathfrak{p}$ equals the ring $A[\lambda_\mathfrak{p}]$.

The extension $K_\mathfrak{p}/K$ is cyclic of degree q^d-1, ramified in $\mathfrak{p}$ and θ^{-1}. It is in fact totally ramified in $\mathfrak{p}$ and the decomposition group at θ^{-1} is isomorphic to the inertia group, therefore isomorphic to $\mathbb{F}_q^\times$. We denote by $\Delta_\mathfrak{p}$ the Galois group $\mathbf{Gal}(K(\Lambda_\mathfrak{p})/K)$. There is an unique isomorphism (Artin symbol, [9, Proposition 7.5.4])

$$\sigma : (A/\mathfrak{p}A)^\times \to \Delta_\mathfrak{p}, \quad \sigma : a \mapsto \sigma_a,$$

such that

$$\sigma_a(\lambda_\mathfrak{p}) = \phi_a(\lambda_\mathfrak{p}).$$

Let $\zeta_1, \ldots, \zeta_d$ be the roots of the polynomial $\mathfrak{p}$ in $\mathbb{F}_{q^d}$. We denote by $\mathbb{F}_\mathfrak{p}$ the field

$$\mathbb{F}_q(\zeta_1, \ldots, \zeta_d).$$

Once chosen a root $\zeta \in \{\zeta_1, \ldots, \zeta_d\}$, the Teichmüller character (see [9, Section 8.11]) $\omega_\mathfrak{p}$ induces an unique group homomorphism

$$\vartheta_\mathfrak{p} : \Delta_\mathfrak{p} \to \mathbb{F}_\mathfrak{p}^\times,$$

defined in the following way: if $\delta = \sigma_a \in \Delta_\mathfrak{p}$ for some $a \in A$, then

$$\vartheta_\mathfrak{p}(\delta) = a(\zeta) = \chi_\zeta(a).$$

We will refer to this homomorphism as to the Techmüller character allowing an abuse of language (indeed, it is customary, in particular, that Teichmüller characters take values in Witt rings).

2.1.1 Gauss-Thakur sums

For any finite abelian group G, we shall write $\widehat{G}$ for the group $\mathbf{Hom}(G,(\mathbb{F}^{\mathrm{alg}})^\times)$. In particular, $\vartheta_\mathfrak{p}\in\widehat{\Delta}_\mathfrak{p}$. For the background on Gauss-Thakur sums we refer to [9, Section 9.8]. In our approach, however, we find it natural to associate Gauss-Thakur sums to elements of $\widehat{\Delta}_\mathfrak{p}$ (compare with loc. cit. Definition 9.8.1).

Definition 7 With $\mathfrak{p}$, d, $\vartheta_\mathfrak{p}$ as above, the *basic Gauss-Thakur sum* $g(\vartheta_\mathfrak{p}^{q^j})$ associated to this data is the element of K^{ab} defined by:

$$g(\vartheta_\mathfrak{p}^{q^j})=\sum_{\delta\in\Delta_\mathfrak{p}}\vartheta_\mathfrak{p}(\delta^{-1})^{q^j}\delta(\lambda_\mathfrak{p})\in\mathbb{F}_\mathfrak{p}[\lambda_\mathfrak{p}].$$

The same sum is denoted by g_j in [9, 21]. The basic Gauss-Thakur sums are be used to define *general Gauss-Thakur sums* associated to arbitrary elements of $\widehat{\Delta}_\mathfrak{p}$. The group $\widehat{\Delta}_\mathfrak{p}$ being isomorphic to $\Delta_\mathfrak{p}$ it is cyclic; it is in fact generated by $\vartheta_\mathfrak{p}$. Let χ be an element of $\widehat{\Delta}_\mathfrak{p}$. There exists an unique integer i with $0<i<q^d$, such that $\chi=\vartheta_\mathfrak{p}^i$. Let us expand i in base q, that is, let us write $i=i_0+i_1q+\cdots+i_{d-1}q^{d-1}$ with $i_j\in\{0,\ldots,d-1\}$. Then, $\chi=\prod_{j=0}^{d-1}(\vartheta_\mathfrak{p}^{q^j})^{i_j}$.

Definition 8 The *general Gauss-Thakur sum* $g(\chi)$ associated to $\chi\in\widehat{\Delta}_\mathfrak{p}$ as above, is defined by:

$$g(\chi)=\prod_{j=0}^{d-1}g(\vartheta_\mathfrak{p}^{q^j})^{i_j}.$$

More generally, let us now consider a non-constant monic polynomial $\mathfrak{a}\in A$. We denote by $\widehat{\Delta}_\mathfrak{a}$ the dual character group $\mathbf{Hom}(\Delta_\mathfrak{a},(\mathbb{F}_q^{\mathrm{alg}})^\times)$. If χ is in $\widehat{\Delta}_\mathfrak{a}$, we set: $\mathbb{F}_q(\chi)=\mathbb{F}_q(\chi(\delta);\delta\in\Delta_\mathfrak{a})\subset\mathbb{F}_q^{\mathrm{alg}}$. We also write

$$\mathbb{F}_\mathfrak{a}=\mathbb{F}_q(\chi;\chi\in\widehat{\Delta}_\mathfrak{a})$$

and we recall that $\mathbf{Gal}(K_\mathfrak{a}(\mathbb{F}_\mathfrak{a})/K(\mathbb{F}_\mathfrak{a}))\simeq\Delta_\mathfrak{a}$. We observe that $\widehat{\Delta}_\mathfrak{a}$ is isomorphic to $\Delta_\mathfrak{a}$ if and only if $\mathfrak{a}$ is squarefree. If $\mathfrak{a}=\mathfrak{p}_1\cdots\mathfrak{p}_n$ with $\mathfrak{p}_1,\ldots,\mathfrak{p}_n$ distinct primes, then

$$\widehat{\Delta}_\mathfrak{a}\simeq\prod_{i=1}^n\widehat{\Delta}_{\mathfrak{p}_i}.$$

Let us then assume that $\mathfrak{a}$ is non-constant and squarefree. We want to extend the definition of the Gauss-Thakur sums to characters in $\widehat{\Delta}_\mathfrak{a}$. For $\chi\in\widehat{\Delta}_\mathfrak{a}$, χ not equal to the trivial character χ_0, there exist r distinct primes $\mathfrak{p}_1,\cdots,\mathfrak{p}_r$ and characters $\chi_1,\ldots,\chi_r$ with $\chi_j\in\widehat{\Delta}_{\mathfrak{p}_j}$ for all j, with

$$\chi=\chi_1\cdots\chi_r.$$

Definition 9 The *Gauss-Thakur sum* associated to χ is the product:

$$g(\chi)=g(\chi_1)\cdots g(\chi_r).$$

The polynomial $\mathfrak{f}_\chi=\mathfrak{p}_1\cdots\mathfrak{p}_r$ is called the *conductor* of χ; it is a divisor of $\mathfrak{a}$. The degree of $\mathfrak{f}_\chi$ will be denoted by d_χ. If $\mathfrak{a}$ itself is a prime $\mathfrak{p}$ of degree d, then $\mathfrak{f}_\chi=\mathfrak{p}$ and $d_\chi=d$.

The following result collects the basic properties of the sums $g(\chi)$ that we need in the sequel, and can be easily deduced from Thakur's results in [21, Theorems I and II].

Proposition 10 *Let $\mathfrak{a} \in A$ be monic, squarefree of degree d. The following properties hold.*

1. *If $\chi = \chi_0$ is the trivial character, then $g(\chi) = 1$.*
2. *For all $\delta \in \Delta_\mathfrak{a}$, we have $\delta(g(\chi)) = \chi(\delta)g(\chi)$.*
3. *If $\chi \neq \chi_0$, then $g(\chi)g(\chi^{-1}) = (-1)^{d_\chi}\mathfrak{f}_\chi$.*

By the normal basis theorem, $K_\mathfrak{a}$ is a free $K[\Delta_\mathfrak{a}]$-module of rank one. Gauss-Thakur's sums allow to determine explicitly a generator of this module:

Lemma 11 *Let us write $\eta_\mathfrak{a} = \sum_{\chi \in \widehat{\Delta}_\mathfrak{a}} g(\chi) \in K_\mathfrak{a}$. Then :*

$$K_\mathfrak{a} = K[\Delta_\mathfrak{a}]\eta_\mathfrak{a},$$

and

$$A_\mathfrak{a} = A[\Delta_\mathfrak{a}]\eta_\mathfrak{a},$$

where $A_\mathfrak{a}$ is the integral closure of A in $K_\mathfrak{a}$.

Moreover, let χ be in $\widehat{\Delta}_\mathfrak{a}$. Then, the following identity holds:

$$K(\mathbb{F}_\mathfrak{a})g(\chi) = \{x \in K_\mathfrak{a}(\mathbb{F}_\mathfrak{a}) \text{ such that for all } \delta \in \Delta_\mathfrak{a}, \delta(x) = \chi(\delta)x\}. \tag{7}$$

Proof. Let us expand $\mathfrak{a}$ in product $\mathfrak{p}_1 \cdots \mathfrak{p}_n$ of distinct primes $\mathfrak{p}_i$. To show that $A_\mathfrak{a} = A[\Delta_\mathfrak{a}]\eta_\mathfrak{a}$ (this yields the identity $K_\mathfrak{a} = K[\Delta_\mathfrak{a}]\eta_\mathfrak{a}$) one sees that

$$A_\mathfrak{a} \simeq A_{\mathfrak{p}_1} \otimes_A \cdots \otimes_A A_{\mathfrak{p}_n},$$

because the discriminants of the extensions $A_{\mathfrak{p}_i}/A$ are pairwise relatively prime and the fields $K_{\mathfrak{p}_i}$ are pairwise linearly disjoint (see [7]). One then uses [3, Théorème 2.5] to conclude with the second identity.

We now prove the identity (7). We recall that if we set, for $\chi \in \widehat{\Delta}_\mathfrak{a}$,

$$e_\chi = \frac{1}{|\Delta_\mathfrak{a}|} \sum_{\delta \in \Delta_\mathfrak{a}} \chi(\delta)\delta^{-1} \in \mathbb{F}_q(\chi)[\Delta_\mathfrak{a}]$$

(well defined because p, the rational prime dividing q, does not divide $|\Delta_\mathfrak{a}|$), then the following identities hold:

- $e_\chi e_\psi = \delta_{\chi,\psi} e_\chi$ (where $\delta_{\chi,\psi}$ denotes Kronecker symbol),
- for all $\delta \in \Delta_\mathfrak{a}$, $\delta e_\chi = \chi(\delta)e_\chi$,
- $\sum_{\chi \in \widehat{\Delta}_\mathfrak{a}} e_\chi = 1$.

This yields $e_\chi \eta_\mathfrak{a} = g(\chi)$. Now, by $K(\mathbb{F}_\mathfrak{a}) = K_\mathfrak{a}(\mathbb{F}_\mathfrak{a})[\Delta_\mathfrak{a}]\eta_\mathfrak{a}$, we get $e_\chi K_\mathfrak{a}(\mathbb{F}_\mathfrak{a}) = K(\mathbb{F}_\mathfrak{a})g(\chi)$. The second part of the Lemma then follows by observing that if M is an $\mathbb{F}_\mathfrak{a}[\Delta_\mathfrak{a}]$-module, then

$$e_\chi M = \{m \in M \text{ such that for all } \delta \in \Delta_\mathfrak{a}, \delta m = \chi(\delta)m\}.$$

□

2.1.2 The function of Anderson and Thakur

For the basic properties of this function, introduced in [1] (see also [15]), we also suggest to read [16, Section 3.1]. We recall from [14, Corollaries 5, 10] that $t \in \mathbb{C}_\infty \setminus \{\theta^{q^k}; k \geq 0\}$ and $\omega(t)$ are simultaneously algebraic if and only if $t = \zeta \in \mathbb{F}_q^{\mathrm{alg}}$. Moreover, we have the following Lemma, where we adopt the convention that $\mathbb{F}_{q^0} := \emptyset$.

Lemma 12 *Let $d \geq 1$ be an integer. For $\zeta \in \mathbb{F}_{q^d} \setminus \mathbb{F}_{q^{d-1}}$, $\omega(\zeta)$ belongs to the set*

$$\mathbb{F}_{q^d}^\times \rho_\zeta,$$

where ρ_ζ is a distinguished root of the polynomial

$$X^{q^d-1} - (\zeta - \theta^{q^{d-1}}) \cdots (\zeta - \theta) \in A[\zeta][X]. \tag{8}$$

Moreover, regardless of the choice of ζ,

$$v_\infty(\omega(\zeta)) = -\frac{1}{q-1}.$$

Proof. This is a simple consequence of [14, Corollary 5]. □

The function of Anderson and Thakur can also be defined, alternatively, by the series expansion:

$$\omega(t) := \sum_{i=0}^\infty \lambda_{\theta^{i+1}} t^i = \sum_{n=0}^\infty \frac{\widetilde{\pi}^{q^n}}{d_n(\theta^{q^n} - t)} \in (-\theta)^{1/(q-1)} K_\infty[[t]] \tag{9}$$

converging for $|t| < q$.

The *Tate algebra* $\mathbb{T}_t$ is the $\mathbb{C}_\infty$-algebra whose elements are the series $\sum_{i\geq 0} c_i t^i \in \mathbb{C}_\infty[[t]]$ converging in the bordered unit disk

$$\overline{D}(0,1) = \{t \in \mathbb{C}_\infty, |t| \leq 1\}.$$

Here, $|\cdot|$ denotes an absolute value associated to the valuation v_∞. In all the following, for clarity, we normalize it by setting $|\theta| = q$.

We recall from [14, Section 4] that the Tate algebra $\mathbb{T}_t$ is endowed with the norm $\|\cdot\|$ defined as follows: if $f = \sum_{i\geq 0} c_i t^i \in \mathbb{T}_t$ with $c_i \in \mathbb{C}_\infty$ $(i \geq 0)$, then $\|f\| = \sup_{i\geq 0} |c_i| = \max_{i\geq 0} |c_i|$. Endowed with this norm, $\mathbb{T}_t$ becomes a $\mathbb{C}_\infty$-Banach algebra. For $r > 0$ a real number, we denote by $\mathbb{D}_r$ the $\mathbb{F}_q[t]$-submodule of $\mathbb{T}_t$ whose elements are the series f such that $\|f\| < r$. The operator τ extends in an unique way to a $\mathbb{F}_q[t]$-automorphism of $\mathbb{T}_t$ so we have at once all the $\mathbb{F}_q[t]$-endomorphisms $\phi_{\mathfrak{a}} - \chi_t(\mathfrak{a})$.

In all the following, we denote by $K[t][\tau]$ and $K[[\tau]]$ respectively the skew polynomial rings in powers of τ with coefficients in $K[t]$ and the skew entire series rings in powers of τ with coefficients in K, endowed with the product rule induced by the identity $\tau t = t\tau$. The evaluation operator (see [14, Section 4])

$$E_{\mathfrak{e}} : \mathbb{C}_\infty \to \mathbb{C}_\infty,$$

where

$$\mathfrak{e} = \sum_{i\geq 0} \frac{\tau^i}{d_i} \in K[[\tau]]$$

is the series associated to Carlitz's exponential function introduced in [14, Section 4], extends to an $\mathbb{F}_q[t]$-endomorphism of $\mathbb{T}_t$. The above formula (9) can be rewritten in a compact form as

$$\omega(t) = E_{\mathfrak{c}}\left(-\frac{\widetilde{\pi}}{t-\theta}\right).$$

We mention that in [16], following a suggestion of D. Goss, an analogy between the above formula and the definition of the gamma function as a Mellin transform of the function e^{-z} was discussed.

From (9), one deduces easily that ω belongs to $\mathbb{T}_t$ (see Subsection 2.2 below). It is also easy to show that ω is a generator of the free $\mathbb{F}_q[t]$-module of rank one, kernel of the operator

$$\phi_\theta - \chi_t(\theta) = \tau + \theta - t \in K[t][\tau],$$

so that

$$\tau\omega(t) = (t-\theta)\omega(t)$$

(see [13, Proposition 3.3.6]). This implies that ω is a generator of the intersection of the kernels in $\mathbb{T}_t$ of the various operators

$$\phi_{\mathfrak{a}} - \chi_t(\mathfrak{a}),$$

with $\mathfrak{a}$ monic as above (see [14, Lemma 29]). We are going to provide, in Proposition 15, a complete description of the kernels of each one of these operators.

2.2 The functions $\omega_{\mathfrak{a},j}$

The following Lemma holds.

Lemma 13 *The kernel of the evaluation operator $E_{\mathfrak{c}}$ is the submodule $\widetilde{\pi}A[t]$ of $\mathbb{T}_t$. Its restriction to $\mathbb{D}_{|\widetilde{\pi}|}$ is an isometry.*

Proof. The kernel clearly contains $\widetilde{\pi}A[t]$. Let m be an element of $\ker(E_{\mathfrak{c}})$. Then, $m = \sum_{i\geq 0} c_i t^i$ with $c_i \in \mathbb{C}_\infty$ and $\exp(c_i) = 0$, so that $c_i \in \widetilde{\pi}A$ for all i. But then,

$$\ker E_{\mathfrak{c}} \subset \widetilde{\pi}A[[t]] \cap \mathbb{T}_t = \widetilde{\pi}A[t].$$

That this endomorphism is an isometry on $\mathbb{D}_{|\widetilde{\pi}|}$ was implicitly observed in [14]. This relies on the fact that exp induces an isometry on the disk $\{z \in \mathbb{C}_\infty; |z| < |\widetilde{\pi}|\}$ and the simple verification is left to the reader. $\square$

In order to define the functions $\omega_{\mathfrak{a},j}$ we will compute, following [14, Section 4], the image of $E_{\mathfrak{c}}$ at various rational functions of $\mathbb{T}_t$, and for this, we will need the next Lemma.

Lemma 14 *Let $\mathfrak{a}$ be a non-constant monic polynomial of A. Then, $1/(\mathfrak{a} - \chi_t(\mathfrak{a})) \in \mathbb{T}_t$.*

Proof. It suffices to show that the roots $\xi \in \mathbb{C}_\infty$ of the polynomial $\mathfrak{a} - \chi_t(\mathfrak{a}) \in A[t]$ are all such that $|\xi| > 1$. But this is obvious, since we have $|\chi_\xi(\mathfrak{a})| > 1$ if and only if $|\xi| > 1$. $\square$

We now fix a monic polynomial $\mathfrak{a}$ of degree $d > 0$. Lemma 14 implies that, for $j = 0, \ldots, d-1$, the series

$$\omega_{\mathfrak{a},j}(t) = E_{\mathfrak{c}}\left(\frac{\theta^j\widetilde{\pi}}{\mathfrak{a} - \chi_t(\mathfrak{a})}\right)$$

are well defined elements of $\mathbb{T}_t$. When the reference to the polynomial $\mathfrak{a}$ is clear, we will write ω_j instead of $\omega_{\mathfrak{a},j}$. In particular, $\omega_{\theta,0} = \omega$. By Lemma 13, we have

$$\|\omega_j\| = \left\| \frac{\theta^j \widetilde{\pi}}{\mathfrak{a} - \chi_t(\mathfrak{a})} \right\| = |\widetilde{\pi}\theta^j a^{-1}| = |\phi_{\theta^j}(\lambda_{\mathfrak{a}})| = q^{\frac{q}{q-1}+j-d}, \quad j = 0, \ldots, d-1. \tag{10}$$

To study the elements ω_j as rigid analytic functions, it may be convenient to observe that the function $\chi_t(\mathfrak{a})$ (with $\mathfrak{a}$ as above) induces a rigid analytic endomorphism of the bordered unit disk $\overline{D}(0,1) = \{t \in \mathbb{C}_\infty; |t| \leq 1\}$. Hence, right composition in series of powers of a new variable x by setting $x = \chi_t(\mathfrak{a})$ induces a map $\mathbb{T}_x \to \mathbb{T}_t$.

Assuming that τ is linearly extended to $\mathbb{T}_x$ by the rule $\tau(x) = x$, let us now consider the series

$$\omega_j^*(x) = \omega_{\mathfrak{a},j}^*(x) = E_{\mathfrak{e}}\left(\frac{\widetilde{\pi}}{\mathfrak{a} - x}\right) = \sum_{n \geq 0} \exp\left(\frac{\widetilde{\pi}}{\mathfrak{a}^{n+1}}\right) x^n \in \mathbb{T}_x, \quad j = 0, \ldots, d-1.$$

Again, $\omega_{\theta,x}^*(x) = \omega(x)$. Furthermore, we notice that, for all $j = 0, \ldots, d-1$, ω_j^* has no zeroes on the disk $\overline{D}(0,1)$. Then,

$$\omega_j(t) = \omega_j^* \circ \chi_t(\mathfrak{a}), \quad j = 0, \ldots, d-1, \tag{11}$$

and we see that these functions have no zeroes in the disk $\overline{D}(0,1)$.

2.2.1 Kernel of the operators $\phi_{\mathfrak{a}} - \chi_t(\mathfrak{a})$

We shall prove the next Proposition.

Proposition 15 *The kernel in $\mathbb{T}_t$ of the operator $\phi_{\mathfrak{a}} - \chi_t(\mathfrak{a})$ is the free $\mathbb{F}_q[t]$-module of rank d generated by the series $\omega_{\mathfrak{a},0}, \ldots, \omega_{\mathfrak{a},d-1}$.*

Proof. We will write ω_j at the place of $\omega_{\mathfrak{a},j}$ for simplicity. Let us consider the column matrix

$$\Omega_{\mathfrak{a}}(t) = \begin{pmatrix} \omega_0 \\ \omega_1 \\ \vdots \\ \omega_{d-1} \end{pmatrix} \in \mathbf{Mat}_{d\times 1}(\mathbb{T}_t).$$

Let us write:

$$\mathfrak{a} = a_0 + a_1\theta + \cdots + a_{d-1}\theta^{d-1} + \theta^d \in A^+, \quad a_0, \ldots, a_{d-1} \in \mathbb{F}_q.$$

By the identity

$$\begin{aligned} E_{\mathfrak{e}}\left(\frac{\theta^d\widetilde{\pi}}{\mathfrak{a} - \chi_t(\mathfrak{a})}\right) &= E_{\mathfrak{e}}\left(\frac{(\mathfrak{a} - \chi_t(\mathfrak{a}) + \chi_t(\mathfrak{a}) - a_0 - a_1\theta - \cdots - a_{d-1}\theta^{d-1})\widetilde{\pi}}{\mathfrak{a} - \chi_t(\mathfrak{a})}\right) \\ &= (\chi_t(\mathfrak{a}) - a_0)\omega_0 - a_1\omega_1 - \cdots - a_{d-1}\omega_{d-1}, \end{aligned}$$

We obtain

$$\phi_\theta \Omega_{\mathfrak{a}} = M_{\mathfrak{a}}(t)\Omega_{\mathfrak{a}},$$

where

$$M_{\mathfrak{a}}(t)=\begin{pmatrix}0 & 1 & 0 & \cdots & 0\\ 0 & 0 & 1 & \cdots & 0\\ \vdots & \vdots & \vdots & & \vdots\\ 0 & 0 & 0 & \cdots & 1\\ -a_0 & -a_1 & -a_2 & \cdots & -a_{d-1}\end{pmatrix}+\begin{pmatrix}0 & 0 & \cdots & 0\\ 0 & 0 & \cdots & 0\\ \vdots & \vdots & & \vdots\\ 0 & 0 & \cdots & 0\\ \chi_t(\mathfrak{a}) & 0 & \cdots & 0\end{pmatrix}\in\mathbf{GL}_d(\mathbb{F}_q(t)),$$

compare with [9, Section 5.3]. Moreover, $M_{\mathfrak{a}}(t)$ commutes with τ, therefore, the matrix $\mathfrak{a}(M_{\mathfrak{a}}(t))$ represents the scalar multiplication by $\chi_t(\mathfrak{a})$ and

$$\phi_{\mathfrak{a}}\Omega_{\mathfrak{a}}=\chi_t(\mathfrak{a})\Omega_{\mathfrak{a}}.$$

This already shows that $\omega_0,\ldots,\omega_{d-1}$ belong to the kernel of $\phi_{\mathfrak{a}}-\chi_t(\mathfrak{a})$.

We now show that these functions are linearly independent over $\mathbb{F}_q[t]$. Let us assume by contradiction that there exist elements $\mu_0,\ldots,\mu_{d-1}\in\mathbb{F}_q[t]$, not all zero, such that

$$\sum_{i=0}^{d-1}\mu_i(t)\omega_i(t)=0. \tag{12}$$

We may even assume, without loss of generality, that there exists a root ζ of $\mathfrak{a}$, and an index $i\in\{0,\ldots,d-1\}$, such that $\mu_i(\zeta)\neq 0$. By (12), we see that

$$E_{\mathfrak{e}}\left(\frac{\widetilde{\pi}}{\mathfrak{a}-\chi_t(\mathfrak{a})}\sum_{i=0}^{d-1}\mu_i(t)\theta^i\right)=0$$

and, by Lemma 13, there exists an element $b\in A[t]$ such that

$$\sum_{i=0}^{d-1}\mu_i(t)\theta^i=b(t)(\mathfrak{a}-\chi_t(\mathfrak{a})).$$

Evaluating at $t=\zeta$ now yields:

$$\sum_{i=0}^{d-1}\mu_i(\zeta)\theta^i=b(\zeta)\mathfrak{a}.$$

The above hypothesis on the μ_i's implies that $b(\zeta)\neq 0$. However, the degrees in θ of the left- and right-hand sides do not agree, in contradiction with our assumption, so that $\omega_0,\ldots,\omega_{d-1}$ are linearly independent over $\mathbb{F}_q[t]$.

To finish the proof of the Proposition, we still need to show that the kernel of the operator $\phi_{\mathfrak{a}}-\chi_t(\mathfrak{a})$ is the free $\mathbb{F}_q[t]$-module of rank d generated by the functions $\omega_0,\ldots,\omega_{d-1}$. Now, $\phi_{\mathfrak{a}}-\chi_t(\mathfrak{a})$ operates on the fraction field $\mathbb{L}_t$ of $\mathbb{T}_t$ as well. The subfield of $\mathbb{L}_t$ of elements which are τ-invariant is equal to $\mathbb{F}_q(t)$. Therefore, the kernel of $\phi_{\mathfrak{a}}-\chi_t(\mathfrak{a})$ is a $\mathbb{F}_q(t)$-vector space of dimension $\leq d$ (because of the Wronskian Lemma). But $\omega_0,\ldots,\omega_{d-1}$ are linearly independent over $\mathbb{F}_q[t]$, hence over $\mathbb{F}_q(t)$, and belong to the kernel which then is equal to the $\mathbb{F}_q(t)$-vector space generated by $\omega_0,\ldots,\omega_{d-1}$.

Let $f\in\mathbb{T}_t$ be such that $\phi_{\mathfrak{a}}(f)-\chi_t(\mathfrak{a})f=0$ and let us consider an element g in $\mathbb{T}_t$ such that $E_{\mathfrak{e}}(g)=f$. By Lemma 13, we have $(a-\chi_t(a))g\in\widetilde{\pi}A[t]$ so that

$$g\in\widetilde{\pi}A[t]+\sum_{j=0}^{d-1}\mathbb{F}_q[t]\frac{\theta^j\widetilde{\pi}}{a-\chi_t(a)}.$$

Therefore,

$$f = E_{\mathfrak{e}}(g) \in \sum_{j=0}^{d-1} \mathbb{F}_q[t]\omega_j(t).$$

$\square$

Remark 16 In analogy with Hölder's Theorem for the gamma function, it is not difficult to show that the functions

$$\omega, \mathcal{D}_1\omega, \ldots, \mathcal{D}_n\omega, \ldots$$

are algebraically independent over $\mathbb{C}_\infty(t)$.

Remark 17 Let n be a nonnegative integer, let us consider a prime $\mathfrak{p} \in A$ of degree $d \geq 1$. It is possible to show, with the same ideas as in the proof of [16, Proposition 15], that the kernel of the operator

$$\phi_{\mathfrak{p}^{n+1}} - \chi_t(\mathfrak{p}^{n+1}),$$

of dimension nd where d is the degree of $\mathfrak{p}$, is spanned by the entries of the matrices

$$\Omega_{\mathfrak{p}}, \mathcal{D}_1\Omega_{\mathfrak{p}}, \ldots, \mathcal{D}_n\Omega_{\mathfrak{p}}.$$

2.3 Analytic identities

In this subsection we proceed to review the main properties of the Tate algebra and higher derivatives we need and then, we describe two families of analytic identities that will be of crucial use in the proof of Theorem 1.

2.3.1 Tate algebras and higher derivatives

Let $t_1, \ldots, t_s$ be independent variables. Extending the observations of the beginning of Section 2.2, we consider now the *Tate algebra* in s variables $\mathbb{T}_{t_1,\ldots,t_s}$, that is, the $\mathbb{C}_\infty$-subalgebra of $\mathbb{C}_\infty[[t_1, \ldots, t_s]]$ whose elements f are formal series

$$f = \sum_{i_1,\ldots,i_s \in \mathbb{Z}_{\geq 0}} f_{i_1,\ldots,i_s} t_1^{i_1} \cdots t_s^{i_s}, \quad f_{i_1,\ldots,i_s} \in \mathbb{C}_\infty \tag{13}$$

converging in the bordered unit polydisk $\overline{D}(0,1)^s$. The algebra $\mathbb{T}_{t_1,\ldots,t_s}$ is endowed with the norm $\|\cdot\|$ generalizing the norm used in Section 2.2 and defined as follows. Let $f \in \mathbb{T}_{t_1,\ldots,t_s}$ as in (13). Then,

$$\|f\| := \sup_{i_1,\ldots,i_s} |f_{i_1,\ldots,i_s}| = \max_{i_1,\ldots,i_s} |f_{i_1,\ldots,i_s}|.$$

We recall that with this norm, $\mathbb{T}_{t_1,\ldots,t_s}$ is a Banach $\mathbb{C}_\infty$-algebra. Furthermore, in this setting, we can extend τ to a $\mathbb{F}_q[t_1, \ldots, t_s]$-automorphism and we have the $\mathbb{F}_q[t_1, \ldots, t_s]$-endomorphism $E_{\mathfrak{e}}$.

It is helpful to also notice that if $f_0, f_1, \ldots, f_s$ are elements of $\mathbb{T}_{t_1,\ldots,t_s}$ with $\|f_i\| \leq 1$ for $i = 1, \ldots, s$, then the composition of functions $f_0(f_1, \ldots, f_s)$ also defines an element of $\mathbb{T}_{t_1,\ldots,t_s}$. Let $\mathbb{T}^{\mathrm{sym}}_{t_1,\ldots,t_d}$ be the sub-algebra of $\mathbb{T}_{t_1,\ldots,t_d}$ of functions which are symmetric in the variables $t_1, \ldots, t_d$. The automorphism τ induces a $\mathbb{F}_q[t_1, \ldots, t_d]^{\mathrm{sym}}$-automorphism of $\mathbb{T}^{\mathrm{sym}}_{t_1,\ldots,t_d}$, where we have denoted by $\mathbb{F}_q[t_1, \ldots, t_d]^{\mathrm{sym}}$ the subring of polynomials which are symmetric in $t_1, \ldots, t_d$.

We will need the following Lemma.

Lemma 18 *Let $\mathfrak{p}$ be a prime of A of degree d, let $\zeta_1, \ldots, \zeta_d$ be its roots in $\mathbb{F}_q^{alg}$, let us consider an element $f \in \mathbb{T}^{sym}_{t_1,\ldots,t_d}$. Then,*

$$(\tau f)(\zeta_1, \ldots, \zeta_d) = f(\zeta_1, \ldots, \zeta_d)^q.$$

Proof. By virtue of [6, Théorème 2, A IV.64], the series of f can be expanded as a series in the elementary symmetric polynomials

$$\alpha_n(t_1, \ldots, t_d) = \sum_{I \subset \{1,\ldots,d\}} \left(\prod_{i \in I} t_i \right), \quad n = 1, \ldots, d$$

(the sum running over the subsets I of $\{1, \ldots, d\}$ of size n), we only need to verify the Lemma for $f = \alpha_n$, $n = 0, \ldots, d$ (indeed, $\mathbb{C}_\infty[t_1, \ldots, t_d]^{\mathrm{sym}}$ is dense for the sup-norm $\|\cdot\|$ in $\mathbb{T}^{\mathrm{sym}}_{t_1,\ldots,t_d}$). But in this case,

$$(\tau \alpha_n)(\zeta_1, \ldots, \zeta_d) = \alpha_n(\zeta_1, \ldots, \zeta_d) = \alpha_n(\zeta_1, \ldots, \zeta_d)^q,$$

because $\zeta_1, \ldots, \zeta_d$ are conjugate. □

Higher derivatives. We will occasionally need to compute the higher derivatives of ω and other allied functions. For the background on higher derivatives (also called *hyperderivatives*), we refer to the recent work of Jeong [12] noticing that the specific tools we are interested in are also contained in Teichmüller's paper [20].

The $\mathbb{C}_\infty$-linear higher derivative $(\mathcal{D}_{t,n})_{n \geq 0}$ (also denoted by $(\mathcal{D}_n)_{n \geq 0}$ in this text) defined by

$$\mathcal{D}_{t,m}(t^n) = \binom{n}{m} t^{n-m}, \quad m, n \geq 0$$

induces $\mathbb{C}_\infty$-linear endomorphisms of $\mathbb{T}_t$. As an example of computation, we have

$$\mathcal{D}_{t,n}(\theta - t)^{-1} = (\theta - t)^{-n-1}, \quad n \geq 0. \tag{14}$$

This higher derivative can be defined over more general $\mathbb{C}_\infty$-algebras of rigid analytic functions of the variable t as well. It also satisfies the chain rule, see [12, Section 2.2] and [20, Equation (6)]. In other words, for all $n \geq 1$, there exist polynomials

$$F_{n,i}(X_1, \ldots, X_{n+1-i}) \in \mathbb{F}_p[X_1, \ldots, X_{n+1-i}], \quad i = 1, \ldots, n$$

(where p is the prime dividing q) with the following property. For f, g rigid analytic functions with g defined over a non-empty open subset $\mathcal{O} \subset \mathbb{C}_\infty$ and $\mathbb{C}_\infty$-valued in such a way that $f \circ g$ is a well defined rigid analytic function $\mathcal{O} \to \mathbb{C}_\infty$,

$$\mathcal{D}_{t,n}(f \circ g) = \sum_{i=1}^{n} F_{n,i}(\mathcal{D}_{t,1} g, \ldots, \mathcal{D}_{t,n+1-i} g)(\mathcal{D}_{t,i} f) \circ g. \tag{15}$$

Moreover, one easily sees that

$$F_{n,n} = X_1^n, \quad F_{n,1} = X_n, \quad n \geq 1.$$

This property holds in particular in $\mathbb{T}_t$ for $f, g \in \mathbb{T}_t$ and $\|g\| \leq 1$, when $f \circ g \in \mathbb{T}_t$. Clearly, for all n, $\mathcal{D}_n$ commutes with the operator $E_{\mathfrak{c}}$. We will often write $\mathcal{D}_n$ instead of $\mathcal{D}_{t,n}$ to simplify our notations. More generally, for all $i = 1, \ldots, s$, $(\mathcal{D}_{t_i,n})_{n \geq 0}$ is a higher derivative of $\mathbb{T}_{t_1,\ldots,t_s}$.

2.3.2 A first family of analytic identities

The next result we need is Proposition 19 below. For $i = 0, \ldots, d-1$, we set:

$$\mathfrak{a}_i = a_{i+1} + a_{i+2}\theta + \cdots + a_{d-1}\theta^{d-i-2} + \theta^{d-i-1},$$

so that, in particular, $\mathfrak{a}_1 = a_1 + a_2\theta + \cdots + a_{d-1}\theta^{d-2} + \theta^{d-1}$, $\mathfrak{a}_{d-2} = a_{d-1} + \theta$, $\mathfrak{a}_{d-1} = 1$. We also set $\mathfrak{a}_{-1} := \mathfrak{a}$ for completeness.

Proposition 19 (First family of analytic identities) *The following identity holds in $\mathbb{T}_t$:*

$$\omega(t) = \sum_{i=0}^{d-1} \chi_t(\mathfrak{a}_i)\omega_i(t). \tag{16}$$

Moreover, for all $n \geq 1$, there exists an element Ω_n of the submodule of $\mathbb{T}_t$

$$\sum_{i=0}^{n-1}\sum_{j=0}^{d-1} \mathbb{F}_q[t](\mathcal{D}_i\omega_j)$$

such that

$$(\mathcal{D}_n\omega)(t) = \sum_{j=0}^{d-1} \chi_t(\mathfrak{a}_j)(\mathcal{D}_n\omega_j)(t) + \Omega_n. \tag{17}$$

Proof. In $\mathbb{F}_q[t,\theta]$, we have the elementary identity:

$$\frac{\mathfrak{a} - \chi_t(\mathfrak{a})}{\theta - t} = \sum_{j=0}^{d-1} \chi_t(\mathfrak{a}_j)\theta^j. \tag{18}$$

Multiplying both sides of it by $\widetilde{\pi}$ and dividing by $\mathfrak{a} - \chi_t(\mathfrak{a})$, we obtain the identity

$$\frac{\widetilde{\pi}}{\theta - t} = \sum_{j=0}^{d-1} \chi_t(\mathfrak{a}_j)\frac{\theta^j\widetilde{\pi}}{\mathfrak{a} - \chi_t(\mathfrak{a})},$$

which holds in $\mathbb{T}_t$ by Lemma 14. The first part of the proposition follows after evaluation of both sides by $E_{\mathfrak{e}}$.

The second part of the proposition is a direct consequence of Leibniz formula for $(\mathcal{D}_n)_{n\geq 0}$, but we give all the details of the intermediate computations for convenience of the reader. Let us consider again (18) and apply $\mathcal{D}_n$ on both left- and right-hand sides. By (14) and the chain rule (Subsection 2.3.1), we verify that

$$\mathcal{D}_n\left(\frac{1}{\mathfrak{a} - \chi_t(\mathfrak{a})}\right) = \frac{\chi_t(\mathfrak{a}')^n}{(\mathfrak{a} - \chi_t(\mathfrak{a}))^{n+1}} + \Xi_n, \tag{19}$$

where, by Leibniz rule, Ξ_n is an element of the submodule of $\mathbb{T}_t$:

$$\sum_{j=0}^{n-1} \mathbb{F}_q[t]\mathcal{D}_j\left(\frac{1}{a - \chi_t(a)}\right). \tag{20}$$

Therefore, by (18),

$$\frac{1}{(\theta-t)^{n+1}}=\sum_{j=0}^{d-1}\theta^j\left(\frac{\chi_t(\mathfrak{a}_j)\chi_t(\mathfrak{a}')^n}{(\mathfrak{a}-\chi_t(\mathfrak{a}))^{n+1}}+\Upsilon_{j,n}\right),$$

where $\Upsilon_{j,n}$ again are elements of the submodule (20). These identities hold in $\mathbb{T}_t$ and since $\mathcal{D}_n$ commutes with $E_{\mathfrak{e}}$, we get, multiplying by $\widetilde{\pi}$ and applying $E_{\mathfrak{e}}$:

$$\begin{aligned}(\mathcal{D}_n\omega)(t) &= \sum_{j=0}^{d-1}\chi_t(\mathfrak{a}_j)\chi_t(\mathfrak{a}')^n E_{\mathfrak{e}}\left(\frac{\widetilde{\pi}\theta^j}{(\mathfrak{a}-\chi_t(\mathfrak{a}))^{n+1}}\right)+\Omega_{j,n}\\ &= \sum_{j=0}^{d-1}\chi_t(\mathfrak{a}_j)\chi_t(\mathfrak{a}')^n(\mathcal{D}_n\omega^*_{\mathfrak{a},j})(\chi_t(\mathfrak{a}))+\Omega_{j,n}\\ &= \sum_{j=0}^{d-1}\chi_t(\mathfrak{a}_j)(\mathcal{D}_n\omega_{\mathfrak{a},j})(t)+\Omega_{j,n}\end{aligned}$$

where $\Omega_{j,n}$ is an element of

$$\mathbb{F}_q[t]\omega_{\mathfrak{a},j}+\cdots+\mathbb{F}_q[t](\mathcal{D}_{n-1}\omega_{\mathfrak{a},j}).$$

$\square$

2.3.3 A second family of analytic identities

Our second family of identities holds in $\mathbb{T}^{\mathrm{sym}}_{t_1,\ldots,t_d}$, for $d\geq 1$. Let us consider the higher derivatives $\mathcal{D}_{t_i,n}$ $(i=1,\ldots,d)$ of

$$R=\mathbb{F}_q[t_1,\ldots,t_d].$$

Their *sum*

$$\partial=(\partial_n)_{n\geq 0}$$

is the family of operators defined by:

$$\partial_n=\sum_{k_1+\cdots+k_d=n}\mathcal{D}_{t_1,k_1}\cdots\mathcal{D}_{t_d,k_d},\quad n\geq 0.$$

It is easy to verify that this also is a higher derivative (to check this, one can use, for example, the multinomial theorem). Their sum also induces a higher derivative of $R^{\mathrm{sym}}(\theta)$, of its completion with respect to the ideal $(t_1,\ldots,t_d)$ (the ring $K[[t_1,\ldots,t_d]]^{\mathrm{sym}}$ of symmetric formal series in powers of $t_1,\ldots,t_d$ with coefficients in K), as well as of $\mathbb{T}^{\mathrm{sym}}_{t_1,\ldots,t_d}$. We also notice that the polynomial $P=\prod_{i=1}^d(\theta-t_i)\in R^{\mathrm{sym}}[\theta]$ is a unit of $\mathbb{T}^{\mathrm{sym}}_{t_1,\ldots,t_d}$. We will need the next elementary Lemma.

Lemma 20 *Let N be a polynomial of $R^{sym}[\theta]$. For all $n\geq 0$, we have, in $\mathbb{T}^{sym}_{t_1,\ldots,t_d}$,*

$$\partial_n\left(\frac{N}{P}\right)=\frac{(\mathcal{D}_{\theta,1}P)^nN}{P^{n+1}}+\Psi_n,$$

where Ψ_n is an element of the module $R^{sym}[\theta]P^{-n}$.

Proof. It is obvious that, for all $j \leq d$, $\partial_n(\theta - t_j)^{-1} = (-1)^n \mathcal{D}_{\theta,n}(\theta - t_j)^{-1}$. Therefore,

$$\partial_n P^{-1} = (-1)^n \mathcal{D}_{\theta,n} P^{-1},$$

so that, by Leibniz's rule and the chain rule,

$$\begin{aligned} \partial_n \left(\frac{N}{P}\right) &= (-1)^n(\mathcal{D}_{\theta,n}P^{-1})N + \Theta_n \\ &= \frac{(\mathcal{D}_{\theta,1}P)^n N}{P^{n+1}} + \Psi_n \end{aligned}$$

(Θ_n is another element of $R^{\mathrm{sym}}[\theta]P^{-n}$). □

For $\mathfrak{a} \in A$ we denote by $U_{\mathfrak{a}}$ the polynomial of $R^{\mathrm{sym}}[\theta]$ such that:

$$\frac{U_{\mathfrak{a}}}{P} = \sum_{i=1}^{d} \frac{\mathfrak{a}(t_i)}{\theta - t_i}. \tag{21}$$

The next Proposition holds:

Proposition 21 (Second family of analytic identities) *The following identity holds in the algebra* $\mathbb{T}^{sym}_{t_1,\ldots,t_d}$*:*

$$\sum_{i=1}^{d} \chi_{t_i}(\mathfrak{a})\omega(t_i) = E_{\mathfrak{e}}\left(\frac{\widetilde{\pi} U_{\mathfrak{a}}}{P}\right). \tag{22}$$

More generally, for all $n \geq 0$*, we have the identity in* $\mathbb{T}^{sym}_{t_1,\ldots,t_d}$*:*

$$\sum_{i=1}^{d} \mathfrak{a}(t_i)(\mathcal{D}_n\omega)(t_i) = E_{\mathfrak{e}}\left(\frac{\widetilde{\pi}(\mathcal{D}_{\theta,1}P)^n U_{\mathfrak{a}}}{P^{n+1}}\right) + \Lambda_n, \tag{23}$$

where Λ_n *is an element of*

$$R^{sym} E_{\mathfrak{e}}\left(\frac{\widetilde{\pi}\mathbb{F}_q[\theta]}{P^n}\right).$$

Proof. The first identity follows by multiplying both sides of (21) by $\widetilde{\pi}$ and applying $E_{\mathfrak{e}}$. We now prove the second identity. For all $n \geq 0$, we have the next identity in $\mathbb{T}^{\mathrm{sym}}_{t_1,\ldots,t_d}$, in virtue of Lemma 20:

$$\partial_n\left(\frac{U_{\mathfrak{a}}}{P}\right) = \frac{(\mathcal{D}_{\theta,1}P)^n U_{\mathfrak{a}}}{P^{n+1}} + \Gamma_n, \tag{24}$$

with Γ_n an element of the module $R^{\mathrm{sym}}[\theta]P^{-n}$. Hence, applying ∂_n on both sides of (21) we get

$$\sum_{i=1}^{d} \frac{a(t_i)}{(\theta - t_i)^{n+1}} = \frac{(\mathcal{D}_{\theta,1}P)^n U_{\mathfrak{a}}}{P^{n+1}} + \Gamma_n + \Sigma_n, \tag{25}$$

where Σ_n is an element of

$$\sum_{i=1}^{d} \mathbb{F}_q[t_i] \frac{1}{(\theta - t_i)^n} \cap \mathbb{T}^{\mathrm{sym}}_{t_1,\ldots,t_d},$$

module which is easily seen to lie inside $R^{\mathrm{sym}}[\theta]P^{-n}$. To conclude, we must apply the operator $E_{\mathfrak{e}}$ on both sides of (25) after having multiplied by $\widetilde{\pi}$. On the left-hand side, we find

$$\sum_{i=1}^{d}\mathfrak{a}(t_i)(\mathcal{D}_n\omega)(t_i).$$

As for the right-hand side, the first term gives the function of $\mathbb{T}^{\mathrm{sym}}_{t_1,\ldots,t_d}$

$$E_{\mathfrak{e}}\left(\frac{\widetilde{\pi}(\mathcal{D}_{\theta,1}P)^nU_{\mathfrak{a}}}{P^{n+1}}\right).$$

The second and third terms, give $E_{\mathfrak{e}}(\Gamma_n+\Sigma_n)$ and yield an element Λ_n of

$$\sum_{i=1}^{n}\mathbb{F}_q[t_1,\ldots,t_d]^{\mathrm{sym}}E_{\mathfrak{e}}\left(\frac{\widetilde{\pi}\mathbb{F}_q[\theta]}{P^i}\right),$$

and the Proposition follows. □

Remark 22 In this remark, not used in our paper, we point out an explicit way to compute the polynomials $U_{\mathfrak{a}}$ by specialization. Let us consider indeterminates $X,X_1,\ldots,X_d,Y_1,\ldots,Y_d$ over a field L. Let us define the polynomials:

$$P=P(X,X_1,\ldots,X_d)=\prod_{i=1}^{d}(X-X_d)$$

and

$$U=U(X,X_1,\ldots,X_d,Y_1,\ldots,Y_d)=\sum_{i=0}^{d-1}(-1)^{d-1-i}\alpha_i^*(X_1,\ldots,X_d,Y_1,\ldots,Y_d)X^i,$$

with

$$\alpha^*_{d-j}(X_1,\ldots,X_d,Y_1,\ldots,Y_d)=\sum{}^{*}Y_{i_1}X_{i_2}\cdots X_{i_j},\quad j=1,\ldots,d$$

where the sum $\sum^*$ is over the j-tuples $(i_1,i_2,\ldots,i_j)$ of pairwise distinct integers $1\leq i_1,\ldots,i_j\leq d$, so that

$$\begin{aligned}\alpha_0^* &= \sum_{j=1}^{d}Y_j\prod_{i\neq j}X_i,\\ \alpha^*_{d-1} &= \sum_{j=1}^{d}Y_j.\end{aligned}$$

For example, if $d=3$, we have

$$\begin{aligned}\alpha_0^* &= X_1X_2Y_3+X_1X_3Y_2+X_2X_3Y_1,\\ \alpha_1^* &= X_1Y_2+X_1Y_3+X_2Y_1+X_2Y_3+X_3Y_1+X_3Y_2,\\ \alpha_2^* &= Y_1+Y_2+Y_3,\end{aligned}$$

and

$$U = \alpha_2^* X^2 - \alpha_1^* X + \alpha_0^*.$$

If σ is a permutation of $\{1, \ldots, d\}$, then

$$\alpha_j^*(X_{\sigma(1)}, \ldots, X_{\sigma(d)}, Y_{\sigma(1)}, \ldots, Y_{\sigma(d)}) = \alpha_j(X_1, \ldots, X_d, Y_1, \ldots, Y_d)$$

for all j (see [6, Exercice 5, A IV.92] for more information). The following formula can be easily proved by induction on $d \geq 1$:

$$\sum_{i=1}^{d} \frac{Y_i}{X - X_i} = \frac{U}{P},$$

so that, for $\mathfrak{a} \in A$ and with the obvious choice of variables:

$$U_{\mathfrak{a}}(\theta, t_1, \ldots, t_d) = U(\theta, t_1, \ldots, t_d, \mathfrak{a}(t_1), \ldots, \mathfrak{a}(t_d)).$$

2.4 Proof of Theorem 1

Let E^∞ be the maximal abelian extension of K tamely ramified at infinity which, by virtue of Hayes result in [11], equals the field

$$\mathbb{F}_q^{\mathrm{alg}}(\lambda_{\mathfrak{a}}; a \in A).$$

Furthermore, let L^∞ be the field

$$\mathbb{F}_q^{\mathrm{alg}}((\mathcal{D}_n \omega)(\zeta); \zeta \in \mathbb{F}_q^{\mathrm{alg}}, n \geq 0),$$

where $\mathcal{D}_n$ denotes the n-th divided derivative with respect to the variable t in $\mathbb{T}_t$ (see Subsection 15). Theorem 1 states that $L^\infty = E^\infty$. Let n be an integer and consider the set $\mathcal{E}_n$ whose elements are the monic polynomials $\mathfrak{a}$ of A such that for all $\mathfrak{p}$ a prime, $\mathfrak{p}^{n+1}$ does not divide $\mathfrak{a}$. Let us define the following subfields of K^{alg}:

$$\begin{aligned} E_n &= \mathbb{F}_q^{\mathrm{alg}}(\phi_b(\lambda_{\mathfrak{a}}); \mathfrak{a} \in \mathcal{E}_n, b \in A) \\ L_n &= \mathbb{F}_q^{\mathrm{alg}}(\omega(\zeta), (\mathcal{D}_1\omega)(\zeta), \ldots, (\mathcal{D}_n\omega)(\zeta); \mathfrak{a}(\zeta) = 0 \text{ for some } \mathfrak{a} \in \mathcal{E}_{n+1}). \end{aligned}$$

We have $E_0 \subset E_1 \subset \cdots \subset E_n \subset \cdots$, $L_0 \subset L_1 \subset \cdots \subset L_n \subset \cdots$, and $\cup_{i=0}^\infty E_i = E^\infty$, $\cup_{i=0}^\infty L_i = L^\infty$. We will prove (see Corollary 27) the identities $E_n = L_n$ by using induction on $n \geq 0$ as the next Proposition indicates; Theorem 1 then follows by a limit process.

The next result constitutes the main step to prove Theorem 1.

Proposition 23 *Let $\mathfrak{p}$ be a prime of degree d, let $\mathbb{F}_{\mathfrak{p}}$ be the extension of $\mathbb{F}_q$ generated by the roots of $\mathfrak{p}$. For all $n \geq 0$, we have the following identity of fields.*

$$\mathbb{F}_{\mathfrak{p}} K_{\mathfrak{p}^{n+1}} = \mathbb{F}_{\mathfrak{p}}((\mathcal{D}_k\omega)(\zeta), \zeta \in \mathbb{F}_{\mathfrak{p}}, 0 \leq k \leq n).$$

The proof of the above Proposition will proceed by induction on $n \geq 0$. Let us denote by $L_{\mathfrak{p},n}$ the field $\mathbb{F}_{\mathfrak{p}}((\mathcal{D}_k\omega)(\zeta), \zeta \in \mathbb{F}_{\mathfrak{p}}, 0 \leq k \leq n)$ and by $E_{\mathfrak{p},n}$ the field $\mathbb{F}_{\mathfrak{p}} K_{\mathfrak{p}^{n+1}}$. We have to show that $L_{\mathfrak{p},n} = E_{\mathfrak{p},n}$ for all $n \geq 0$. To ease the reading we will consider the case $n = 0$ separately, before considering the general case, although this discrimination of the two cases is not strictly necessary.

To prove the case $n = 0$, we will use the following Lemma.

Lemma 24 *Let $\mathfrak{p}$ be a prime of degree d, let $\zeta_1,\ldots,\zeta_d \in \mathbb{F}_{\mathfrak{p}}$ be its roots. Let us also consider a polynomial $a \in A$. Then,*

$$\sum_{j=1}^{d} a(\zeta_j)\omega(\zeta_j) = \phi_{a\mathfrak{p}'}(\lambda_{\mathfrak{p}}).$$

Proof. Compare with [16, Proposition 29]. We recall that, in the Tate algebra $\mathbb{T}^{\mathrm{sym}}_{t_1,\ldots,t_d}$, we have, by the identity (22):

$$\sum_{i=1}^{d} a(t_i)\omega(t_i) = E_{\mathfrak{e}}\left(\frac{\widetilde{\pi}U_a}{P}\right),$$

with $P = \prod_{i=1}^{d}(\theta - t_i)$ and U_a as in (21). If we replace $t_i = \zeta_i$ $(i = 1,\ldots,d)$ conjugate elements of $\mathbb{F}_{q^d}$, we have $P(\theta,\zeta_1,\ldots,\zeta_d) = \mathfrak{p}$, $U_a(\theta,\zeta_1,\ldots,\zeta_d) \equiv \mathfrak{p}'a \pmod{\mathfrak{p}}$ and the Lemma follows applying Lemma 18 because:

$$E_{\mathfrak{e}}\left(\frac{\widetilde{\pi}U_a}{P}\right)\bigg|_{t_i=\zeta_i} = \exp\left(\frac{\widetilde{\pi}\mathfrak{p}'a}{\mathfrak{p}}\right) = \phi_{a\mathfrak{p}'}(\lambda_{\mathfrak{p}})$$

(remember that U_a and P are polynomials which are symmetric in $t_1,\ldots,t_d$). □

Proof of Proposition 23 in the case $n = 0$. We begin by applying the first part of Proposition 19, with $\mathfrak{a} = \mathfrak{p}$. In this case we get, for $\zeta \in \mathbb{F}_{\mathfrak{p}}$ a root of $\mathfrak{p}$, $\omega(\zeta) \in \mathbb{F}_{\mathfrak{p}}(\omega^*_{\mathfrak{p},0}(0),\ldots,\omega^*_{\mathfrak{p},d-1}(0)) = E_{\mathfrak{p},0}$ (by (11)), from which the inclusion of fields $L_{\mathfrak{p},0} \subset E_{\mathfrak{p},0}$ follows. For the reverse inclusion we notice that $E_{\mathfrak{p},0} = \mathbb{F}_{\mathfrak{p}}(\lambda_{\mathfrak{p}}) = \mathbb{F}_{\mathfrak{p}}(\phi_{\mathfrak{p}'}(\lambda_{\mathfrak{p}}))$ because $\mathfrak{p}$ and $\mathfrak{p}'$ are relatively prime. By Lemma 24 with $a = 1$, we see that $\phi_{\mathfrak{p}'}(\lambda_{\mathfrak{p}}) \in L_{\mathfrak{p},0}$. Therefore, $L_{\mathfrak{p},0} = E_{\mathfrak{p},0}$. □

To prove the case $n > 0$ of the Proposition 23, we will use two Lemmas. The next Lemma is about the evaluation of the higher derivatives of the functions $\omega_{\mathfrak{a},j}$ at roots of unity.

Lemma 25 *Let $\mathfrak{p}$ be a prime of degree d and ζ a root of $\mathfrak{p}$. Let $n \geq 1$ and $0 \leq j \leq d-1$ be integers. Then, there exists an element $\mu_n \in E_{\mathfrak{p},n-1}$ such that*

$$(\mathcal{D}_n\omega_{\mathfrak{p},j})(\zeta) = (\mathfrak{p}'(\zeta))^n\phi_{\theta^j}(\lambda_{\mathfrak{p}^{n+1}}) + \mu_n.$$

Proof. By (11) and the chain rule, $(\mathcal{D}_n\omega_{\mathfrak{p},j})(\zeta)$ is equal to $(\mathfrak{p}'(\zeta))^n(\mathcal{D}_{x,n}\omega^*_{\mathfrak{p},j})(0)$ plus a linear combination of values $(\mathcal{D}_{x,i}\omega^*_{\mathfrak{p},j})(0)$ with $i = 0,\ldots,n-1$ and coefficients in $\mathbb{F}_{\mathfrak{p}}$. □

Lemma 26 *Let $\mathfrak{p}$ be a prime of degree d, let $\zeta_1,\ldots,\zeta_d \in \mathbb{F}_{\mathfrak{p}}$ be its roots, and let us consider an integer $n \geq 1$. Furthermore, let us also consider a polynomial $a \in A$. Then, there exist an element $\nu_n \in E_{\mathfrak{p},n-1}$ such that*

$$\sum_{j=1}^{d} a(\zeta_j)(\mathcal{D}_n\omega)(\zeta_j) = \phi_{a(\mathfrak{p}')^{n+1}}(\lambda_{\mathfrak{p}^{n+1}}) + \nu_n.$$

Proof. By (23) and Lemma 18, we have

$$\sum_{i=1}^{d} a(\zeta_i)(\mathcal{D}_n\omega)(\zeta_i) = \exp\left(\frac{(\mathfrak{p}')^{n+1}a}{\mathfrak{p}^{n+1}}\right) + \Lambda_n(\zeta_1,\ldots,\zeta_d).$$

The Lemma follows by setting $\nu_n = \Lambda_n(\zeta_1, \ldots, \zeta_d)$, belonging to $E_{\mathfrak{p},n-1} = L_{\mathfrak{p},n-1}$. Indeed, $\exp((\mathfrak{p}')^{n+1}a/\mathfrak{p}^{n+1}) = \phi_{(\mathfrak{p}')^{n+1}a}(\lambda_{\mathfrak{p}^{n+1}})$. $\square$

Proof of Proposition 23 in the case $n \geq 0$. We proceed by induction on n; the case $n = 0$ being already proved, let us assume that $n > 0$ and that $E_{\mathfrak{p},i} = L_{\mathfrak{p},i}$ for $i = 0, \ldots, n-1$. We first show that $L_{\mathfrak{p},n}$ is contained in $E_{\mathfrak{p},n}$. To do so, it suffices to show that $(\mathcal{D}_n\omega)(\zeta) \in E_{\mathfrak{p},n}$ for all ζ root of $\mathfrak{p}$.

By (17) with $t = \zeta$, we have

$$(\mathcal{D}_n\omega)(\zeta) = \sum_{j=0}^{d-1} \mathfrak{p}_j(\zeta)(\mathcal{D}_n\omega_{\mathfrak{p},j})(\zeta) + \Omega_n(\zeta),$$

where it is obvious that $\Omega_n(\zeta) \in L_{\mathfrak{p},n-1} = E_{\mathfrak{p},n-1}$. By Lemma 25, the sum over j equals

$$\sum_{j=0}^{d-1} \mathfrak{p}_j(\zeta)\mathfrak{p}'(\zeta)^n \phi_{\theta^j}(\lambda_{\mathfrak{p}^{n+1}}) + \mu_n,$$

with $\mu_n \in E_{\mathfrak{p},n-1}$. This shows the inclusion $L_{\mathfrak{p},n} \subset E_{\mathfrak{p},n}$.

Let us now show the opposite inclusion. Since $\mathfrak{p}$ is prime, the class of $(\mathfrak{p}')^{n+1}$ is a generator of the A-module $A/\mathfrak{p}^{n+1}$. Therefore, in view of Lemma 26, we will only need to show that $\phi_{(\mathfrak{p}')^{n+1}}(\lambda_{\mathfrak{p}^{n+1}}) \in L_{\mathfrak{p},n}$. But ν_n is element of $E_{\mathfrak{p},n-1} = L_{\mathfrak{p},n-1}$ by hypothesis, so that Lemma 26 implies that $E_{\mathfrak{p},n} \subset L_{\mathfrak{p},n}$. The proof of Proposition 23 is complete. $\square$

Corollary 27 *We have, for all $n \geq 0$, $E_n = L_n$.*

Proof. On one side, E_n is the compositum of all the fields $E_{\mathfrak{p},n}$ with $\mathfrak{p}$ varying in the set of primes of A. On the other side, L_n is the compositum of all the fields $L_{\mathfrak{p},n}$, with $\mathfrak{p}$ varying in the set of primes of A. The result follows from Proposition 23. $\square$

Theorem 1 follows at once by taking $n \to \infty$. $\square$

Remark 28 Theorem 1 does not seem to be directly related to Anderson's result in [2], where he proves that the compositum of all the subfields of $\mathbb{Q}^{\mathrm{alg}}$ that are at once quadratic over $\mathbb{Q}^{\mathrm{ab}}$ and Galois over $\mathbb{Q}$ is generated by *algebraic Γ-monomials* satisfying the Koblitz-Ogus condition. However, an analogue of our result concerning the Akhiezer-Baker function $\Gamma(s-t)$ as in [16] and its composition with $t =$meromorphic integral-periodic functions should hold and will, we hope, be object of further investigation.

2.5 Proof of Theorem 3

By the second part of Proposition 10, we have that for $\delta \in \Delta_{\mathfrak{p}}$ and $\chi \in \widehat{\Delta}_{\mathfrak{p}}$,

$$\delta(g(\chi)) = \chi(\delta)g(\chi). \tag{26}$$

Let ζ be the root of $\mathfrak{p}$ such that $\vartheta_{\mathfrak{p}}(\sigma_\theta)=\zeta$. We have, for $j=0,\ldots,d-1$:

$$\begin{aligned}
g(\vartheta_{\mathfrak{p}}^{q^j}) &= -\sum_{\delta\in\Delta_{\mathfrak{p}}}\vartheta_{\mathfrak{p}}(\delta^{-1})^{q^j}\delta(\lambda_{\mathfrak{p}})\\
&= -\sum_{a\in(A/\mathfrak{p}A)^\times}\vartheta_{\mathfrak{p}}(\sigma_a)^{-q^j}\sigma_a(\lambda_{\mathfrak{p}})\\
&= -\sum_{a\in(A/\mathfrak{p}A)^\times}\vartheta_{\mathfrak{p}}(\sigma_a)^{-q^j}\phi_a(\lambda_{\mathfrak{p}}).
\end{aligned}$$

By (26) and Lemma 24 ($\zeta_1,\ldots,\zeta_d$ again denote the zeros of $\mathfrak{p}$):

$$\begin{aligned}
\chi_\zeta(\mathfrak{p}')^{q^j}g(\vartheta_{\mathfrak{p}}^{q^j}) &= \sigma_{\mathfrak{p}'}(g(\vartheta_{\mathfrak{p}}^{q^j}))\\
&= -\sum_{a\in(A/\mathfrak{p}A)^\times}\vartheta_{\mathfrak{p}}(\sigma_{a\mathfrak{p}'})^{-q^j}\phi_{a\mathfrak{p}'}(\lambda_{\mathfrak{p}})\\
&= -\sum_{a\in(A/\mathfrak{p}A)^\times}\vartheta_{\mathfrak{p}}(\sigma_{a\mathfrak{p}'})^{-q^j}\sum_{k=1}^{d}a(\zeta_k)\omega(\zeta_k)\\
&= -\sum_{i=0}^{d-1}\omega(\zeta^{q^i})\sum_{a\in(A/\mathfrak{p}A)^\times}\vartheta_{\mathfrak{p}}(\sigma_a)^{q^i-q^j},
\end{aligned}$$

where the last identity follows from the fact that $\{\zeta_1,\ldots,\zeta_d\}=\{\zeta,\zeta^q,\ldots,\zeta^{q^{d-1}}\}$. Now, the sum

$$\sum_{a\in(A/\mathfrak{p}A)^\times}\vartheta_{\mathfrak{p}}(\sigma_a)^{q^i-q^j}$$

always vanishes except when $i=j$, case in which the sum equals -1. Therefore:

$$\omega(\zeta^{q^j})=\omega(\vartheta_{\mathfrak{p}}(\sigma_\theta)^{q^j})=\chi_\zeta(\mathfrak{p}')^{q^j}g(\vartheta_{\mathfrak{p}}^{q^j})$$

hence completing the proof of the Theorem. □

We shall also mention the following result.

Corollary 29 *Let ζ be the root of $\mathfrak{p}$ such that $\vartheta_{\mathfrak{p}}(\sigma_\theta)=\zeta$. The following identity holds:*

$$\mathbb{F}_{\mathfrak{p}}K_{\mathfrak{p}}=\mathbb{F}_{\mathfrak{p}}(g(\vartheta_{\mathfrak{p}})).$$

Proof. By Proposition 23, we have $\mathbb{F}_{\mathfrak{p}}K_{\mathfrak{p}}=\mathbb{F}_{\mathfrak{p}}(\omega(\zeta))$. Theorem 3 now implies that $\mathbb{F}_{\mathfrak{p}}(\omega(\zeta))=\mathbb{F}_{\mathfrak{p}}(g(\vartheta_{\mathfrak{p}}))$. □

3 Functional identities for L-series

In this section, we prove Theorem 4. We will need a few preliminary results that we shall study in this subsection. Let d,s be non-negative integers. We begin with the study of the vanishing of the sums

$$S_{d,s}=S_{d,s}(t_1,\ldots,t_s)=\sum_{a\in A^+(d)}\chi_{t_1}(a)\cdots\chi_{t_s}(a)\in\mathbb{F}_q[t_1,\ldots,t_s],$$

which are symmetric polynomials in $t_1, \ldots, t_s$ of total degree $\le ds$, with the standard conventions on empty products. We recall that, for $n \ge 0$,

$$\sum_{a \in \mathbb{F}_q} a^n$$

equals -1 if $n \equiv 0 \pmod{q-1}$ and $n \ge 1$, and equals 0 otherwise. We owe the next Lemma to D. Simon [5]. We give the proof here for the sake of completeness.

Lemma 30 (Simon's Lemma) *We have $S_{d,s} \neq 0$ if and only if $d(q-1) \le s$.*

Proof. Since

$$S_{d,s} = \sum_{a_0 \in \mathbb{F}_q} \cdots \sum_{a_{d-1} \in \mathbb{F}_q} \prod_{i=1}^{s} (a_0 + a_1 t_i + \cdots + a_{d-1} t_i^{d-1} + t_i^d),$$

the coefficient $c_{v_1,\ldots,v_s}$ of $t_1^{v_1} \cdots t_s^{v_s}$ with $v_i \le d$ $(i = 1, \ldots, s)$ is given by the sum:

$$\sum_{a_0 \in \mathbb{F}_q} \cdots \sum_{a_{d-1} \in \mathbb{F}_q} a_{v_1} \cdots a_{v_s},$$

if we set $a_d = 1$. The last sum can be rewritten as:

$$c_{v_1,\ldots,v_s} = \left(\sum_{a_0 \in \mathbb{F}_q} a_0^{\mu_0} \right) \cdots \left(\sum_{a_{d-1} \in \mathbb{F}_q} a_{d-1}^{\mu_{d-1}} \right), \tag{27}$$

where μ_i is the cardinality of the set of the indices j such that $v_j = i$, from which one notices that

$$\sum_{i=0}^{d-1} \mu_i \le s$$

(notice also that $s - \sum_i \mu_i$ is the cardinality of the set of indices j such that $v_j = d$). For any choice of $\mu_0, \ldots, \mu_{d-1}$ such that $\sum_i \mu_i \le s$, there exists $(v_1, \ldots, v_s)$ such that (27) holds.

If $s < d(q-1)$, for all $(v_1, \ldots, v_s)$ as above, there exists i such that, in (27), $\mu_i < q-1$ so that $S_{d,s} = 0$. On the other hand, if $s \ge d(q-1)$, it is certainly possible to find $(v_1, \ldots, v_s)$ such that, in (27), $\mu_0 = \cdots = \mu_{d-1} = q-1$ so that the sum does not vanish in this case. □

As an immediate corollary of Lemma 30, we see that the series

$$F_s = F_s(t_1, \ldots, t_s) = \sum_{d \ge 0} S_{d,s} = \sum_{d \ge 0} \sum_{a \in A^+(d)} \chi_{t_1}(a) \cdots \chi_{t_s}(a)$$

defines a symmetric polynomial of $\mathbb{F}_q[t_1, \ldots, t_s]$ of total degree at most $\frac{s^2}{q-1}$. In the next Lemma, we provide a sufficient condition for the vanishing of the polynomial F_s.

Lemma 31 *If $s \ge 1$, then, $F_s = 0$ if and only if $s \equiv 0 \pmod{q-1}$.*

Proof. Let us assume first that $s \equiv 0 \pmod{q-1}$. The hypothesis on s implies that

$$\sum_{a\in A,\deg_\theta(a)=d} \chi_{t_1}(a)\cdots\chi_{t_s}(a) = -S_{d,s}.$$

We denote by $A(\leq d)$ the set of polynomials of A of degree $\leq d$ and we write

$$G_{d,s} = \sum_{a\in A(\leq d)} \chi_{t_1}(a)\cdots\chi_{t_s}(a).$$

We then have:

$$G_{\frac{s}{q-1},s} = -F_s.$$

Let us choose now distinct primes $\mathfrak{p}_1,\ldots,\mathfrak{p}_s$ of respective degrees $d_1,\ldots,d_s \geq s/(q-1)$ and $\mathfrak{f} = \mathfrak{p}_1\cdots\mathfrak{p}_s$. For all $i = 1,\ldots,s$, we choose a root $\zeta_i \in \mathbb{F}_q^{\mathrm{alg}}$ of $\mathfrak{p}_i$. Let us then consider the Dirichlet character of the first kind $\chi = \chi_{\zeta_1}\cdots\chi_{\zeta_s}$. We have:

$$\begin{aligned}
F_s(\zeta_1,\ldots,\zeta_s) &= -G_{\frac{s}{q-1},s}(\zeta_1,\ldots,\zeta_s)\\
&= -\sum_{a\in A(\leq s/(q-1))}\chi(a)\\
&= -\sum_{a\in A(\leq d_1+\cdots+d_s)}\chi(a)\\
&= -\sum_{a\in (A/\mathfrak{f}A)^\times}\chi(a)\\
&= 0,
\end{aligned}$$

by [19, Proposition 15.3]. Since the set of s-tuples $(\zeta_1,\ldots,\zeta_s)\in(\mathbb{F}_q^{\mathrm{alg}})^s$ with $\zeta_1,\ldots,\zeta_s$ as above is Zariski-dense in $\mathbb{A}^s(\mathbb{C}_\infty)$, this implies the vanishing of F_s. On the other hand, if $s \not\equiv 0 \pmod{q-1}$, then $F_s(\theta,\ldots,\theta) = \zeta(-s)$ the s-th "odd negative" Goss' zeta value which is non-zero, see [8].

□

3.1 Analyticity

The functions $L(\chi_{t_1}\cdots\chi_{t_s},\alpha)$ are in fact rigid analytic entire functions of s variables. This property, mentioned in [14], can be deduced from the more general Proposition 32 that we give here for convenience of the reader.

Let a be a monic polynomial of A. we set:

$$\langle a\rangle = \frac{a}{\theta^{\deg_\theta(a)}} \in 1+\theta^{-1}\mathbb{F}_q[\theta^{-1}].$$

Let $y\in\mathbb{Z}_p$, where p is the prime dividing q. Since $\langle a\rangle$ is a 1-unit of K_∞, we can consider its exponentiation by y:

$$\langle a\rangle^y = \sum_{j\geq 0}\binom{y}{j}(\langle a\rangle-1)^j \in \mathbb{F}_q[[\theta^{-1}]].$$

Here, the binomial $\binom{y}{j}$ is defined, for j a non-negative integer, by extending Lucas formula: writing the p-adic expansion $\sum_{i\geq 0} y_i p^i$ of y ($y_i \in \{0,\ldots,p-1\}$) and the p-adic expansion $\sum_{i=0}^{r} j_i p^i$ of j ($j_i \in \{0,\ldots,p-1\}$), we are explicitly setting:

$$\binom{y}{j} = \prod_{i=0}^{r} \binom{y_i}{j_i}.$$

We also recall, from [9, Chapter 8], the topological group $\mathbb{S}_\infty = \mathbb{C}_\infty^\times \times \mathbb{Z}_p$. For $(x,y) \in \mathbb{S}_\infty$ and d, s non-negative integers, we define the sum

$$S_{d,s}(x,y) = S_{d,s}(x,y)(t_1,\ldots,t_s) = x^{-d} \sum_{a\in A^+(d)} \chi_{t_1}(a)\cdots\chi_{t_s}(a)\langle a\rangle^y \in x^{-d}K_\infty[t_1,\ldots,t_s],$$

which is, for all x, y, a symmetric polynomial of total degree $\leq ds$.

Let us further define, more generally, for variables $t_1,\ldots,t_s \in \mathbb{C}_\infty$ and $(x,y) \in \mathbb{S}_\infty$, the series:

$$L(\chi_{t_1}\ldots\chi_{t_s}; x, y) = \sum_{d\geq 0} S_{d,s}(x,y)(t_1,\ldots,t_s).$$

For fixed choices of $(x,y) \in \mathbb{S}_\infty$, it is easy to show that

$$L(\chi_{t_1}\ldots\chi_{t_s}; x, y) \in \mathbb{C}_\infty[[t_1,\ldots,t_s]],$$

and with a little additional work, one also verifies that this series defines an element of $\mathbb{T}_{t_1,\ldots,t_s}$. Of course, if $(x,y) = (\theta^\alpha, -\alpha)$ with $\alpha > 0$ integer, we find

$$L(\chi_{t_1}\ldots\chi_{t_s}; \theta^\alpha, -\alpha) = L(\chi_{t_1}\ldots\chi_{t_s}, \alpha).$$

The next Proposition holds and improves results of Goss; see [10, Theorems 1, 2]).

Proposition 32 *The series $L(\chi_{t_1},\ldots,\chi_{t_s}; x, y)$ converges for all $(t_1,\ldots,t_s)$ and for all $(x,y) \in \mathbb{S}_\infty$, to a continuous-analytic function on $\mathbb{C}_\infty^s \times \mathbb{S}_\infty$ in the sense of Goss.*

The proof of this result is a simple consequence of the Lemma below. The norm $\|\cdot\|$ used in the Lemma is that of $\mathbb{T}_{t_1,\ldots,t_s}$.

Lemma 33 *Let (x,y) be in $\mathbb{S}_\infty$ and let us consider an integer $d > (s+1)/(q-1)$, with $s > 0$. Then:*

$$\|S_{d,s}(x,y)\| \leq |x|^{-d} q^{-q^{\lfloor d - \frac{s+1}{q-1}\rfloor}}.$$

Proof. Let us write the p-adic expansion $y = \sum_{n\geq 0} c_n p^n$, with $c_n \in \{0,\ldots,p-1\}$ for all n. Collecting blocks of e consecutive terms (where $q = p^e$), this yields a "q-adic" expansion, from which we can extract partial sums:

$$y_n = \sum_{k=0}^{en-1} c_k p^k = \sum_{i=0}^{n-1} u_i q^i \in \mathbb{Z}_{\geq 0},$$

where

$$u_i = \sum_{j=ei}^{e(i+1)-1} c_j p^{j-ei} \in \{0,\ldots,q-1\}.$$

In particular, for $n \geq 0$, we observe that $\ell_q(y_n) \leq n(q-1)$. Since

$$\begin{aligned} S_{d,s}(x,y_n) &= \frac{1}{x^d\theta^{dy_n}}\sum_{a\in A^+(d)}\chi_{t_1}(a)\cdots\chi_{t_s}(a)a^{y_n} \\ &= \frac{1}{x^d\theta^{dy_n}}S_{d,r}(t_1,\ldots,t_s,\underbrace{\theta,\ldots,\theta}_{u_0\text{ times}},\underbrace{\theta^q,\ldots,\theta^q}_{u_1\text{ times}},\ldots,\underbrace{\theta^{q^{n-1}},\ldots,\theta^{q^{n-1}}}_{u_{n-1}\text{ times}}) \end{aligned}$$

with $r = s+\ell_q(y_n)$, if $d(q-1) > s+\ell_q(y_n)$, we have by Simon's Lemma 30:

$$S_{d,s}(x,y_n) = 0.$$

This condition is ensured if $d(q-1) > s+n(q-1)$.

Now, we claim that

$$\|S_{d,s}(x,y)-S_{d,s}(x,y_n)\| \leq |x|^{-d}q^{-q^n}.$$

Indeed,

$$S_{d,s}(x,y)-S_{d,s}(x,y_n) = x^{-d}\sum_{a\in A^+(d)}\chi_{t_1}(a)\cdots\chi_{t_s}(a)\sum_{j\geq 0}\left(\binom{y}{j}-\binom{y_n}{j}\right)(\langle a\rangle-1)^j,$$

and $\binom{y}{j} = \binom{y_n}{j}$ for $j = 0,\ldots,q^n-1$ by Lucas' formula and the definition of the binomial, so that

$$\left|\sum_{j\geq 0}\left(\binom{y}{j}-\binom{y_n}{j}\right)(\langle a\rangle-1)^j\right| \leq q^{-q^n}.$$

The Lemma follows by choosing $n = \lfloor d-1-\frac{s+1}{q-1}\rfloor$. □

In particular, we have the following Corollary to Proposition 32 which generalizes [10, Theorem 1], the deduction of which, easy, is left to the reader.

Corollary 34 *For any choice of an integer $\alpha > 0$ and non-negative integers $M_1,\ldots,M_s$, the function*

$$L(\chi_{t_1}^{M_1}\cdots\chi_{t_s}^{M_s},\alpha) = \sum_{d\geq 0}\sum_{a\in A^+(d)}\chi_{t_1}(a)^{M_1}\cdots\chi_{t_s}(a)^{M_s}a^{-\alpha}$$

defines a rigid analytic entire function $\mathbb{C}_\infty^s \to \mathbb{C}_\infty$.

3.2 Computation of polynomials with coefficients in K_∞

Lemma 35 *For all $d \geq 0$, we have:*

$$S_d(-\alpha) = \sum_{a\in A^+(d)} a^{-\alpha} \neq 0.$$

Proof. This follows from [9, proof of Lemma 8.24.13]. □

We introduce, for d, s, α nonnegative integers, the sum:

$$S_{d,s}(-\alpha) = \sum_{a \in A^+(d)} \chi_{t_1}(s) \cdots \chi_{t_s}(a) a^{-\alpha} \in K[t_1, \ldots, t_s],$$

representing a symmetric polynomial of $K[t_1, \ldots, t_s]$ of exact total degree ds by Lemma 35. We have, with the notations of Section 3.1:

$$S_{d,s}(-\alpha) = S_{d,s}(\theta^\alpha, -\alpha).$$

From the above results, we deduce the following Proposition.

Proposition 36 *Let $l \geq 0$ be an integer such that $q^l - \alpha \geq 0$ and $2 \leq \ell_q(q^l - \alpha) + s \leq d(q-1)$. Then:*

$$S_{d,s}(-\alpha) \equiv 0 \pmod{\prod_{j=1}^{s}(t_j - \theta^{q^l})}.$$

Furthermore, assume that $s \equiv \alpha \pmod{q-1}$. With l as above, let k be an integer such that $k(q-1) \geq \ell_q(q^l - \alpha) + s$. Then:

$$\sum_{d=0}^{k} S_{d,s}(-\alpha) \equiv 0 \pmod{\prod_{j=1}^{s}(t_j - \theta^{q^l})}.$$

Proof. Let us write $m = \ell_q(q^l - \alpha)$. We have $s - 1 + m < d(q-1)$ so that, by Simon's Lemma 30, $S_{d,s-1+m} = 0$. Now, let us write the q-ary expansion $q^l - \alpha = n_0 + n_1 q + \cdots + n_r q^r$ with $n_i \in \{0, \ldots, q-1\}$ and let us observe that, since $q^l - \alpha \geq 0$,

$$\begin{aligned}
S_{d,s}(-\alpha)(t_1, \ldots, t_{s-1}, \theta^{q^l}) &= \sum_{a \in A^+(d)} \chi_{t_1}(a) \cdots \chi_{t_{s-1}}(a) a^{q^l - \alpha} \\
&= \sum_{a \in A^+(d)} \chi_{t_1}(a) \cdots \chi_{t_{s-1}}(a) \chi_\theta(a)^{n_0} \chi_{\theta^q}(a)^{n_1} \cdots \chi_{\theta^{q^r}}(a)^{n_r} \\
&= S_{d,s-1+m}(t_1, \ldots, t_{s-1}, \underbrace{\theta, \ldots, \theta}_{n_0 \text{ times}}, \underbrace{\theta^q, \ldots, \theta^q}_{n_1 \text{ times}}, \ldots, \underbrace{\theta^{q^r}, \ldots, \theta^{q^r}}_{n_r \text{ times}}) \\
&= 0.
\end{aligned}$$

Therefore $t_s - \theta^{q^l}$ divides $S_{d,s}(-\alpha)$. The first part of the Proposition follows from the fact that this polynomial is symmetric. For the second part, we notice by the first part, that the condition on k is sufficient for the sum $S_{d,s}(-\alpha)(t_1, \ldots, t_s)$ to be congruent modulo $(t_s - \theta^{q^l})$ for all $d \geq k+1$. On the other hand, by Lemma 31 and the above computation, we have

$$\sum_{d \geq 0} S_{d,s}(-\alpha)(t_1, \ldots, t_{s-1}, \theta^{q^l}) = F_s(t_1, \ldots, t_{s-1}, \theta, \ldots, \theta^{q^r}) = 0.$$

But then, thanks to the condition on k,

$$\sum_{d=0}^{k} S_{d,s}(-\alpha) \equiv -\sum_{d > k} S_{d,s}(-\alpha) \equiv 0 \pmod{(t_s - \theta^{q^l})}$$

and the Proposition follows again because the sum we are inspecting is a symmetric polynomial. $\square$

We further have the result below.

Proposition 37 *Let $s,\alpha \geq 1$, $s \equiv \alpha \pmod{q-1}$. Let δ be the smallest positive integer such that $q^\delta \geq \alpha$ and $s + \ell_q(q^\delta - \alpha) \geq 2$. Then, the function of Theorem 4*

$$V_{\alpha,s}(t_1,\ldots,t_s) = L(\chi_{t_1}\cdots\chi_{t_s},\alpha)\omega(t_1)\cdots\omega(t_s)\widetilde{\pi}^{-\alpha}\left(\prod_{i=1}^{s}\prod_{j=0}^{\delta-1}\left(1-\frac{t_i}{\theta^{q^j}}\right)\right)$$

is in fact a symmetric polynomial of $K_\infty[t_1,\ldots,t_s]$. Moreover, its total degree $\delta(\alpha,s)$ is not bigger than $s\left(\frac{s+\ell_q(q^\delta-\alpha)}{q-1}\right) - s$.

Proof. Let δ be the smallest positive integer such that $q^\delta - \alpha \geq 0$ and $s + \ell_q(q^\delta - \alpha) \geq 2$. We fix an integer k such that

$$k(q-1) \geq s + \ell_q(q^\delta - \alpha). \tag{28}$$

We also set:

$$N(k) = \delta + k - \frac{s+\ell_q(q^\delta-\alpha)}{q-1}.$$

Obviously, $N(k) \geq \delta$. Let l be an integer such that

$$\delta \leq l \leq N(k).$$

We claim that we also have

$$k(q-1) \geq s + \ell_q(q^l - \alpha).$$

Indeed, let us write the q-ary expansion $\alpha = \alpha_0 + \alpha_1 q + \cdots + \alpha_m q^m$ with $\alpha_m \neq 0$. Then, $\delta = m$ if $\alpha = q^m$ and $s \geq 2$ and $\delta = m+1$ otherwise. If l is now an integer $l \geq \delta$, we have

$$\begin{aligned} q^l - \alpha &= q^l - q^\delta + q^\delta - \alpha \\ &= q^\delta(q-1)\left(\sum_{i=0}^{l-\delta-1} q^i\right) + q^\delta - \alpha, \end{aligned}$$

where the sum over i is zero if $l = \delta$, and

$$\ell_q(q^l - \alpha) = (q-1)(l-\delta) + \ell_q(q^\delta - \alpha)$$

because there is no carry over in the above sum. Now, the claim follows from (28).

By Proposition 36 we have, with k as above, that the following expression

$$W_{k,s,\alpha} := \left(\prod_{i=1}^{s}\prod_{j=\delta}^{N(k)}\left(1-\frac{t_i}{\theta^{q^j}}\right)^{-1}\right)\sum_{d=0}^{k} S_{d,s}(-\alpha)$$

is in fact a symmetric polynomial in $K[t_1,\ldots,t_s]$. By Lemma 35, $S_{d,s}(-\alpha)\in K[t_1,\ldots,t_s]$ is a symmetric polynomial of total degree ds; indeed, the coefficient of $t_1^d\cdots t_s^d$ is exactly $S_d(-\alpha)$. Hence, the total degree of $\sum_{d=0}^k S_{d,s}(-\alpha)$ is exactly ks. The total degree of the product

$$\prod_{i=1}^{s}\prod_{j=\delta}^{N(k)}\left(1-\frac{t_i}{\theta^{q^j}}\right)$$

is equal to $s(1+N(k)-\delta)$ so that, by the definition of $N(k)$:

$$\begin{aligned}\deg(W_{k,s,\alpha}) &= sk-s-sN(k)+s\delta\\ &= sk-sk-s\delta+s\delta-s+s\left(\frac{s+\ell_q(q^\delta-\alpha)}{q-1}\right)\\ &= s\left(\frac{s+\ell_q(q^\delta-\alpha)}{q-1}\right)-s,\end{aligned}$$

independent on k. We now let k tend to infinity. The Proposition follows directly from the definition (2) of ω as an infinite product, the fact that, in (2), $\widetilde{\pi}\theta^{-1/(q-1)}\in K_\infty$, and the definition of $L(\chi_{t_1}\cdots\chi_{t_s},\alpha)$. □

3.3 An intermediate result on special values of Goss L-functions

Let χ be a Dirichlet character of the first kind, that is, a character

$$\chi:(A/\mathfrak{a}A)^\times\to(\mathbb{F}_q^{\mathrm{alg}})^\times,$$

where $\mathfrak{a}$ is a non-constant squarefree monic element of A which we identify, by abuse of notation, to a character of $\widehat{\Delta}_{\mathfrak{a}}$ still denoted by χ, of conductor $\mathfrak{f}=\mathfrak{f}_\chi$, and degree $d=\deg_\theta f$.

Let $s(\chi)$ be the *type* of χ, that is, the unique integer $s(\chi)\in\{0,\ldots,q-2\}$ such that:

$$\chi(\zeta)=\zeta^{s(\chi)}\quad\text{for all }\zeta\in\mathbb{F}_q^\times.$$

We now consider the *generalized α-th Bernoulli number* $B_{\alpha,\chi^{-1}}\in\mathbb{F}_q(\chi)(\theta)$ associated to χ^{-1}, [4, Section 2], and the special value of Goss' *abelian L-function* [9, Section 8]:

$$L(\alpha,\chi)=\sum_{a\in A^+}\chi(a)a^{-\alpha},\quad\alpha\geq 1.$$

The following result is inspired by the proof of [4, Proposition 8.2]:

Proposition 38 *Let $\alpha\geq1$, $\alpha\equiv s(\chi)\pmod{q-1}$. Then:*

$$\frac{L(\alpha,\chi)g(\chi)}{\widetilde{\pi}^\alpha}=(-1)^d\frac{B_{\alpha,\chi^{-1}}}{\mathfrak{f}^{\alpha-1}}\in\mathbb{F}_q(\chi)(\theta).$$

Proof. The proposition is known to be true for the trivial character (see [9, Section 9.2]); in this case, we notice that:

$$B_{\alpha,\chi_0^{-1}}=\frac{BC_\alpha}{\Pi(\alpha)},\quad\alpha\geq1,\quad\alpha\equiv0\pmod{q-1},$$

where we recall that BC_α is the α-th Bernoulli-Carlitz number and $\Pi(\alpha)$ is the Carlitz factorial of α (see [9, Definition 9.2.1]). We now assume that $\chi \neq \chi_0$. Since:

$$\exp(z) = z \prod_{a \in A \setminus \{0\}} \left(1 - \frac{z}{\widetilde{\pi} a}\right),$$

We have:

$$\frac{1}{\exp(z)} = \sum_{a \in A} \frac{1}{z - \widetilde{\pi} a}.$$

Let $b \in A$ be relatively prime with $\mathfrak{f}$ and let $\sigma_b \in \mathbf{Gal}(K_\mathfrak{f}/K)$ be the element such that $\sigma_b(\lambda_\mathfrak{f}) = \phi_b(\lambda_\mathfrak{f})$. We have:

$$\frac{1}{\exp(z) - \sigma_b(\lambda_\mathfrak{f})} = -\sum_{n \geq 0} \frac{\mathfrak{f}^{n+1}}{\widetilde{\pi}^{n+1}} \left(\sum_{a \in A} \frac{1}{(b + a\mathfrak{f})^{n+1}} \right) z^n.$$

Therefore, we obtain:

$$\sum_{b \in (A/\mathfrak{f}A)^\times} \frac{\chi(b)}{\exp(z) - \sigma_b(\lambda_\mathfrak{f})} = -\sum_{n \geq 0} \frac{\mathfrak{f}^{n+1}}{\widetilde{\pi}^{n+1}} \left(\sum_{a \in A \setminus \{0\}} \frac{\chi(a)}{a^{n+1}} \right) z^n.$$

If $n + 1 \not\equiv s(\chi) \pmod{q-1}$, we get:

$$\sum_{a \in A \setminus \{0\}} \frac{\chi(a)}{a^{n+1}} = 0,$$

and if $n + 1 \equiv s(\chi) \pmod{q-1}$, we have:

$$\sum_{a \in A \setminus \{0\}} \frac{\chi(a)}{a^{n+1}} = -L(n+1, \chi).$$

Thus:

$$\sum_{b \in (A/\mathfrak{f}A)^\times} \frac{\chi(b)}{\exp(z) - \sigma_b(\lambda_\mathfrak{f})} = \sum_{i \geq 1,\, i \equiv s(\chi) \pmod{(q-1)}} \frac{\mathfrak{f}^i L(i, \chi)}{\widetilde{\pi}^i} z^{i-1}. \tag{29}$$

But note that by the second part of Lemma 11:

$$\sum_{b \in (A/\mathfrak{f}A)^\times} \frac{\chi(b)}{\exp(z) - \sigma_b(\lambda_\mathfrak{f})} \in g(\chi^{-1}) F_q(\chi)(\theta)[[z]].$$

Since by Proposition 10,

$$g(\chi) g(\chi^{-1}) = (-1)^d \mathfrak{f},$$

where $d = \deg_\theta f_\chi$, we get the result by comparison of the coefficients of the series expansion of both sides of (29). $\square$

Remark 39 In the above proof of Proposition 38, if we set $\alpha = 1$ we have, by comparison of the constant terms in the series expansions in powers of z in (29):

$$\widetilde{\pi}^{-1}\mathfrak{f}L(1,\chi) = -\sum_{b\in(A/\mathfrak{f}A)^\times}\frac{\chi(b)}{\sigma_b(\lambda_\mathfrak{f})} \in g(\chi^{-1})\mathbb{F}_q(\chi)(\theta).$$

Assuming that $\mathfrak{f}$ is not a prime, by [19, Proposition 12.6], $\lambda_\mathfrak{f}$ is a unit in the integral closure $A_\mathfrak{f}$ of A in $K_\mathfrak{f}$. Therefore,

$$\sum_{b\in(A/\mathfrak{f}A)^\times}\frac{\chi(b)}{\sigma_b(\lambda_\mathfrak{f})} \in g(\chi^{-1})\mathbb{F}_q(\chi)[\theta]$$

and we deduce that

$$\widetilde{\pi}^{-1}L(1,\chi)g(\chi) \in \mathbb{F}_q(\chi)[\theta].$$

This remark will be crucial in the proof of Corollary 41.

3.4 Proof of Theorem 4

The next Lemma provides a rationality criterion for a polynomial a priori with coefficients in K_∞, again based on evaluation at roots of unity.

Lemma 40 *Let $F(t_1,\ldots,t_s) \in K_\infty[t_1,\ldots,t_s]$ such that for all $\zeta_1,\ldots,\zeta_s \in \mathbb{F}_q^{alg}$, pairwise not conjugate over $\mathbb{F}_q$,*

$$F(\zeta_1,\ldots,\zeta_s) \in K(\zeta_1,\ldots,\zeta_s).$$

Then $F(t_1,\ldots,t_s) \in K[t_1,\ldots,t_s]$.

Proof. We begin by pointing out that if elements $a_1,\ldots,a_s \in K_\infty$ are $K\otimes_{\mathbb{F}_q}\mathbb{F}_q^{\mathrm{alg}}$-linearly dependent, then they also are K-linearly dependent. The proof proceeds by induction on $s \geq 1$. For $s = 1$, this is obvious. Now, let

$$\sum_{i=1}^{s}\lambda_s a_s = 0 \tag{30}$$

be a non-trivial relation of linear dependence with the $\lambda_i \in K\otimes\mathbb{F}_q^{\mathrm{alg}}\setminus\{0\}$. We may assume that $\lambda_s = 1$ and that there exists $i \in \{1,\ldots,s-1\}$ such that $\lambda_i \notin K$. Then, there exists

$$\sigma \in \mathbf{Gal}(K_\infty\otimes\mathbb{F}_q^{\mathrm{alg}}/K_\infty) = \mathbf{Gal}(K\otimes\mathbb{F}_q^{\mathrm{alg}}/K) = \mathbf{Gal}(\mathbb{F}_q^{\mathrm{alg}}/\mathbb{F}_q)$$

such that $\sigma(\lambda_i) \neq \lambda_i$. Applying σ on both left- and right-hand sides of (30) and subtracting, yields a non-trivial relation involving at most $s-1$ elements of K_∞ on which we can apply the induction hypothesis.

We can now complete the proof of the Lemma. Let F be a polynomial in $K_\infty[t_1,\ldots,t_s]$ not in $K[t_1,\ldots,t_s]$. It is easy to show that there exist $a_1,\ldots,a_m \in K_\infty$, linearly independent over K, such that

$$F = a_1P_1 + \cdots + a_mP_m,$$

where $P_1,\ldots,P_m$ are non-zero polynomials of $K[t_1,\ldots,t_s]$. Let us suppose by contradiction that there exists $F \in K_\infty[t_1,\ldots,t_s]\setminus K[t_1,\ldots,t_s]$ satisfying the hypotheses of the Lemma. Since the set of s-tuples $(\zeta_1,\ldots,\zeta_s)$ as in the statement of the Lemma is Zariski-dense in $\mathbb{A}^s(\mathbb{C}_\infty)$, there

exist a choice of such roots of unit $\zeta_1, \ldots, \zeta_s$ and $i \in \{1, \ldots, m\}$ such that $P_i(\zeta_1, \ldots, \zeta_m) \neq 0$. This means that $a_1, \ldots, a_m$ are $K \otimes \mathbb{F}_q^{\mathrm{alg}}$-linearly dependent, thus K-linearly dependent by the previous observations; a contradiction. □

Proof of Theorem 4. In view of Lemma 40, we want to show that the polynomial

$$V_{\alpha,s} = \widetilde{\pi}^{-\alpha} L(\chi_{t_1} \cdots \chi_{t_s}, \alpha)\omega(t_1)\cdots\omega(t_s)\left(\prod_{i=1}^{s}\prod_{j=0}^{\delta-1}\left(1 - \frac{t_i}{\theta^{q^j}}\right)\right) \in K_\infty[t_1, \ldots, t_s]$$

of Proposition 37 takes values in $K(\zeta_1, \ldots, \zeta_s)$ for all $\zeta_1, \ldots, \zeta_s \in \mathbb{F}_q^{\mathrm{alg}}$ pairwise non conjugate over $\mathbb{F}_q$. Let $(\zeta_1, \ldots, \zeta_s)$ be one of such s-tuples of roots of unity and, for $i = 1, \ldots, s$, let $\mathfrak{p}_i \in A$ be the minimal polynomial of ζ_i, so that $\mathfrak{p}_1, \ldots, \mathfrak{p}_s$ are pairwise relatively prime. We choose the characters $\vartheta_{\mathfrak{p}_i}$ so that $\vartheta_{\mathfrak{p}_i}(\sigma_\theta) = \zeta_i$ for all i. We construct the Dirichlet character of the first kind χ defined, for $a \in A$, by

$$\chi(a) = \chi_{\zeta_1}(a) \cdots \chi_{\zeta_s}(a).$$

By Proposition 38, we have

$$\frac{L(\alpha, \chi) g(\chi)}{\widetilde{\pi}^\alpha} = (-1)^{d_\chi} \frac{B_{\alpha,\chi^{-1}}}{\mathfrak{f}_\chi^{\alpha-1}} \in \mathbb{F}_q(\chi)(\theta).$$

Since

$$L(\alpha, \chi) = L(\chi_{\zeta_1} \cdots \chi_{\zeta_s}, \alpha),$$

we get:

$$\begin{aligned} V_{\alpha,s}(\zeta_1, \ldots, \zeta_s) &= L(\alpha, \chi)\omega(\zeta_1)\cdots\omega(\zeta_s)\widetilde{\pi}^{-\alpha} \\ &= \frac{L(\alpha, \chi)g(\chi)}{\widetilde{\pi}^\alpha}\frac{\omega(\zeta_1)\cdots\omega(\zeta_s)}{g(\chi)} \\ &= (-1)^{d_\chi}\frac{B_{\alpha,\chi^{-1}}}{\mathfrak{f}_\chi^{\alpha-1}}\chi_{\zeta_1}(\mathfrak{p}_1')\cdots\chi_{\zeta_s}(\mathfrak{p}_s') \\ &\in K(\zeta_1, \ldots, \zeta_s), \end{aligned}$$

where in the next to last step, we have used Theorem 3. The proof of Theorem 4 now follows from Lemma 40. □

4 Congruences for Bernoulli-Carlitz numbers

In this Section, we shall prove Theorem 5. This is possible because in Theorem 4, more can be said when $\alpha = 1$. In this case, one sees that the integer δ of Theorem 4 is equal to zero and $s \geq q$, so that, with the notations of that result,

$$V_{1,s} = \widetilde{\pi}^{-1} L(\chi_{t_1} \cdots \chi_{t_s}, 1)\omega(t_1)\cdots\omega(t_s).$$

In the next Subsection we will show that this is a polynomial of $A[t_1, \ldots, t_s]$.

4.1 Functional identities with $\alpha = 1$

We begin with the following Corollary of Theorem 4. The main result of this subsection is Proposition 44.

Corollary 41 *Let $s \geq 2$ be such that $s \equiv 1 \pmod{q-1}$. Then the symmetric polynomial $V_{1,s} \in K[t_1, \ldots, t_s]$ of Theorem 4 is in fact a polynomial of $F_q[\theta][t_1, \ldots, t_s]$ of total degree $\leq s^2/(q-1) - s$ in the variables $t_1, \ldots, t_s$.*

Proof. It follows from a simple modification of the proof of Proposition 38. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_s$ be distinct primes in A, let us write $\mathfrak{a} = \mathfrak{p}_1 \cdots \mathfrak{p}_s$ and let us consider the Dirichlet character χ associated to $\vartheta_{\mathfrak{p}_1} \cdots \vartheta_{\mathfrak{p}_s}$ that we also loosely identify with the corresponding element of $\widehat{\Delta}_{\mathfrak{a}}$. Since $\mathfrak{a}$ is not a prime power, Remark 39 implies that

$$\widetilde{\pi}^{-1} L(1, \chi) g(\chi) \in \mathbb{F}_q(\chi)[\theta]. \tag{31}$$

Now, specializing at $t_i = \zeta_i$ the root of $\mathfrak{p}_i$ associated to the choice of characters $\vartheta_{\mathfrak{p}_i}$ for all $i = 1, \ldots, s$, we obtain

$$V_{1,s}(\zeta_1, \ldots, \zeta_s) = \widetilde{\pi}^{-1} L(1, \chi) g(\chi) \in \mathbb{F}_q(\zeta_1, \ldots, \zeta_s)[\theta],$$

and the result follows from Lemma 40, the bound on the degree agreeing with that of Theorem 4. □

4.1.1 Digit principle for the function ω and the L-series

Let $\varphi : \mathbb{T}_t \to \mathbb{T}_t$ be the $\mathbb{C}_\infty$-linear map defined by

$$\varphi\left(\sum_{n \geq 0} c_n t^n\right) = \sum_{n \geq 0} c_n t^{qn}, \quad c_n \in \mathbb{C}_\infty.$$

We also set, for N a non-negative integer with its expansion in base q, $N = N_0 + N_1 q + \cdots + N_r q^r$, $N_i \in \{0, \ldots, q-1\}$:

$$\omega_N(X) = \prod_{i=0}^{r} \varphi^i(\omega(X))^{N_i}.$$

We then have the next Lemma.

Lemma 42 *The following identity holds:*

$$\omega_N(\vartheta_{\mathfrak{p}}(\sigma_\theta)) = \vartheta_{\mathfrak{p}}(\sigma_{\mathfrak{p}'})^N g(\vartheta_{\mathfrak{p}}^N).$$

Proof. This is a direct application of Theorem 3. Indeed,

$$\omega_N(\vartheta_{\mathfrak{p}}(\sigma_\theta)) = \prod_{i=0}^{d-1} \omega(\vartheta_{\mathfrak{p}}(\sigma_{\theta^{q^i}}))^{N_i} = \prod_{i=0}^{d-1} \vartheta_{\mathfrak{p}}(\sigma_{\mathfrak{p}'})^{q^i N_i} g(\vartheta_{\mathfrak{p}}^{q^i})^{N_i}.$$

□

Let X, Y be two indeterminates over K. We introduce a family of polynomials $(G_d)_{d \geq 0}$ in $\mathbb{F}_q[X, Y]$ as follows. We set $G_0(X, Y) = 1$ and

$$G_d(X, Y) = \prod_{i=0}^{d-1} (X - Y^{q^i}), \quad d \geq 1.$$

This sequence is closely related to the sequence of polynomials $G_n(y)$ of [1, Section 3.6]: indeed, the latter can be rewritten in terms of the former:

$$G_d(y) = G_d(T^{q^d}, y^q), \quad d \geq 1,$$

in both notations of loc. cit. and ours ([4]). The polynomial G_d is monic of degree d in the variable X, and $(-1)^d G_d$ is monic in the variable Y of degree $(q^d-1)/(q-1)$. We now define, for $N = N_0 + N_1 q + \cdots + N_r q^r$ a non-negative integer expanded in base q, the polynomial

$$H_N(t) = \prod_{i=0}^{r} G_i(t^{q^i}, \theta)^{N_i} = \prod_{i=0}^{r} \prod_{j=0}^{i-1} (t^{q^i} - \theta^{q^j})^{N_i}.$$

We also define the quantities associated to N and q:

$$\begin{aligned} \mu_q(N) &= \sum_{i=0}^{r} N_i i q^i, \\ \mu_q^*(N) &= \frac{N}{q-1} - \frac{\ell_q(N)}{q-1}, \\ \ell'_q(N) &= \sum_{i=0}^{r} N_i i. \end{aligned}$$

Lemma 43 *Let N be a non-negative integer. The following properties hold.*

1. *The polynomial $H_N(t)$, as a polynomial of the indeterminate t, is monic of degree $\mu_q(N)$.*
2. *As a polynomial of the indeterminate θ, $H_N(t)$ has degree $\mu_q^*(N)$ and the leading coefficient is $(-1)^{\ell'_q(N)}$.*
3. *We have $H_N(\theta) = \Pi(N)$ and $v_\infty(H_N(\theta)) = \mu_q(N)$, where v_∞ is the ∞-adic valuation of $\mathbb{C}_\infty$.*
4. *We also have, for all $\zeta \in \mathbb{F}_q^{alg}$, $v_\infty(H_N(\zeta)) = -\mu_q^*(N)$.*

Proof. Easy and left to the reader. □

We observe that:

$$\varphi^d \omega(t) = \frac{1}{G_d(t^{q^d}, \theta)} \omega(t)^{q^d} = \omega_{q^d N}(t), \quad d \geq 0$$

[4] As an aside remark, we also notice that we recover in this way the coefficients of the formal series in $K[[\tau]]$ associated to Carlitz's exponential and logarithm

$$\mathfrak{e} = \sum_{i \geq 0} d_i^{-1} \tau^i, \quad \mathfrak{l} = \sum_{i \geq 0} l_i^{-1} \tau^i,$$

because $d_i = G_i(\theta^{q^i}, \theta)$ and $l_i = G_i(\theta, \theta^q)$. Moreover, if $\mathfrak{p}$ is a prime of A of degree d, we observe that

$$\mathfrak{p} = \prod_{i=1}^{d} (\theta - \zeta_i) = \prod_{j=0}^{d-1} (\theta - \vartheta_{\mathfrak{p}}(\sigma_{\theta q^j})) = G_d(\theta, \vartheta_{\mathfrak{p}}(\sigma_\theta)).$$

so that, with N as above,

$$\omega_N(t) = \frac{\omega(t)^N}{\prod_{i=0}^r G_i(t^{q^i},\theta)^{N_i}} = \frac{\omega(t)^N}{H_N(t)}. \tag{32}$$

The following Proposition was inspired by a discussion with D. Goss.

Proposition 44 *Let $s \geq 2$ be an integer. Let $M_1, \ldots, M_s$ be positive integers such that $M_1 + \cdots + M_s \equiv 1 \pmod{q-1}$. Then:*

$$W(t_1,\ldots,t_s) = \widetilde{\pi}^{-1} L(\chi_{t_1}^{M_1}\cdots\chi_{t_s}^{M_s},1)\omega_{M_1}(t_1)\cdots\omega_{M_s}(t_s) \in \mathbb{F}_q[\theta,t_1,\ldots,t_s].$$

For all i, the degree in t_i of W satisfies

$$\deg_{t_i}(W) \leq M_i\left(\frac{\sum_j M_j}{q-1} - 1\right) - \mu_q(M_i)$$

Proof. We shall write

$$H = \prod_{i=1}^s H_{M_i}(t_i).$$

We know from Lemma 43 that $\deg_{t_i}(H) = \mu_q(M_i)$. Let us consider the function

$$V = \widetilde{\pi}^{-1} L(\chi_{t_1}^{M_1}\cdots\chi_{t_s}^{M_s},1)\omega^{M_1}(t_1)\cdots\omega^{M_s}(t_s),$$

so that by (32),

$$V = WH.$$

Corollary 41 implies that:

$$V \in \mathbb{F}_q[\theta,t_1,\cdots,t_s]$$

and we are done if we can prove that H divides V in $\mathbb{F}_q[\theta,t_1,\ldots,t_s]$.

Let $\mathfrak{p}_1,\ldots,\mathfrak{p}_s$ be distinct primes of A such that $|\mathfrak{p}_i| - 1 > M_i$, and let $\zeta_1,\ldots,\zeta_s$ be respective roots of these polynomials chosen in compatibility with the characters $\vartheta_{\mathfrak{p}_1},\ldots,\vartheta_{\mathfrak{p}_s}$. Let us also write

$$\chi = \vartheta_{\mathfrak{p}_1}^{M_1}\cdots\vartheta_{\mathfrak{p}_s}^{M_s}.$$

By Lemma 42,

$$\omega_{M_1}(\zeta_1)\cdots\omega_{M_s}(\zeta_s) = \vartheta_{\mathfrak{p}_1}(\sigma_{\mathfrak{p}_1'})^{M_1}\cdots\vartheta_{\mathfrak{p}_s}(\sigma_{\mathfrak{p}_s'})^{M_s} g(\chi).$$

Therefore,

$$W(\zeta_1,\ldots,\zeta_s) = \widetilde{\pi}^{-1}L(1,\chi)g(\chi)\vartheta_{\mathfrak{p}_1}(\sigma_{\mathfrak{p}_1'})^{M_1}\cdots\vartheta_{\mathfrak{p}_s}(\sigma_{\mathfrak{p}_s'})^{M_s}.$$

By (31), $\widetilde{\pi}^{-1}L(1,\chi)g(\chi) \in \mathbb{F}_q(\chi)[\theta]$, while $\prod_{i=1}^s \vartheta_{\mathfrak{p}_i}(\sigma_{\mathfrak{p}_i'})^{M_i} \in \mathbb{F}_q(\chi)$ so that

$$W(\zeta_1,\cdots,\zeta_s) \in \mathbb{F}_q(\chi)[\theta] = \mathbb{F}_q(\zeta_1,\ldots,\zeta_s)[\theta].$$

Now, H is a polynomial in θ with leading coefficient in $\mathbb{F}_q^\times$ (see Lemma 43). Dividing V by H as polynomials in θ we find

$$V = HQ + R,$$

where Q, R are polynomials in $\mathbb{F}_q[\theta,t_1,\ldots,t_s]$, and $\deg_\theta R < \deg_\theta H = \sum_i \mu_q^*(M_i)$ (the last inequality by Lemma 43). But for $\zeta_1,\ldots,\zeta_s$ as above, we must have $Q(\theta,\zeta_1,\ldots,\zeta_s) = W(\zeta_1,\cdots,\zeta_s)$ and

$$R(\zeta_1,\ldots,\zeta_s) = 0.$$

This implies $R = 0$ and thus $W = Q \in \mathbb{F}_q[\theta,t_1,\ldots,t_s]$. □

4.1.2 The polynomials W_s

By Proposition 44, the function

$$W_s(t) = \widetilde{\pi}^{-1} L(\chi_t^s, 1)\omega_N(t) = \frac{L(\chi_t^s, 1)\omega(t)^s}{\widetilde{\pi} H_s(t)}$$

is a polynomial of $\mathbb{F}_q[t, \theta]$. Furthermore, we have:

Proposition 45 *Assuming that $s \geq 2$ is an integer congruent to 1 modulo $q-1$ and is not a power of q, the following properties hold.*

1. *The degree in t of W_s does not exceed $s(s-1)/(q-1) - s - \mu_q(s)$,*
2. *the degree in θ of W_s is equal to $(\ell_q(s) - q)/(q-1)$.*

By the remarks in the introduction, we know how to handle the case of $s = q^i$; we then have

$$W_{q^i}(t) = \frac{1}{\theta - t^{q^i}}.$$

Proof of Proposition 45. The bound for the degree in t is a simple consequence of Proposition 44 and Lemma 43. To show the property of the degree in θ, we first notice that, by Lemma 43, for all $\zeta \in \mathbb{F}_q^{\mathrm{alg}}$,

$$v_\infty(W_s(\zeta)) = -\frac{\ell_q(s) - q}{q-1}. \tag{33}$$

The computation of $W_s(\zeta)$ is even explicit if $\zeta \in \mathbb{F}_q$. Indeed, with the appropriate choice of a $(q-1)$-th root of $(\zeta - \theta)$, the fact that $\chi_\zeta = \chi_\zeta^s$, Lemma 12 and [14, Theorem 1],

$$\begin{aligned} W_s(\zeta) &= \frac{L(\chi_\zeta^s, 1)\omega(\zeta)^s}{\widetilde{\pi} H_s(\zeta)} \\ &= \frac{L(\chi_\zeta, 1)\omega(\zeta)^s}{\widetilde{\pi} H_s(\zeta)} \\ &= \frac{L(\chi_\zeta, 1)\omega(\zeta)^s}{\widetilde{\pi}(\zeta - \theta)^{\frac{s-\ell_q(s)}{q-1}}} \\ &= (\zeta - \theta)^{-\frac{1}{q-1}}(\theta - \zeta)^{-1}(\zeta - \theta)^{\frac{s}{q-1}}(\zeta - \theta)^{\frac{\ell_q(s)-s}{q-1}} \end{aligned}$$

and

$$W_s(\zeta) = -(\zeta - \theta)^{\frac{\ell_q(s)-q}{q-1}}. \tag{34}$$

Let us write:

$$W_s(t) = \sum_{i=0}^{g} a_i t^i, \quad a_i \in A.$$

By (34), we have

$$a_0 = W_s(0) = -(-\theta)^{\frac{\ell_q(s)-q}{q-1}} \tag{35}$$

and for all $\zeta \in \mathbb{F}_q^{\mathrm{alg}}$ we have, by (33),

$$|W_s(\zeta)| = |a_0|.$$

This means that for $i = 1, \ldots, g$, $|a_i| < |a_0|$, and the identity on the degree in θ follows as well. $\square$

Corollary 46 *If $\ell_q(s) = q$, then $W_s = -1$.*

Proof. By (34), $W_s = a_0 = -1$ in virtue of (35). $\square$

By Corollary 41, the function

$$V_{1,s}(t_1,\ldots,t_s) = \widetilde{\pi}^{-1} L(\chi_{t_1}\cdots\chi_{t_s})\omega(t_1)\cdots\omega(t_s)$$

is, for $s \equiv 1 \pmod{q-1}$ and $s \geq 2$, a polynomial of $A[t_1,\ldots,t_s]$. Since

$$\omega(t) = \frac{\widetilde{\pi}}{\theta - t} + o(1),$$

where $o(1)$ represents a function locally analytic at $t = \theta$, the function $L(\chi_{t_1}\cdots\chi_{t_s}, 1)$ vanishes on the divisor

$$D = \bigcup_{i=1}^{s} D_i,$$

where

$$D_i = \{(t_1,\ldots,t_{i-1},\theta,t_{i+1},\ldots,t_s) \in \mathbb{C}_\infty\}.$$

In other words, in $\mathbb{C}_\infty[[t_1-\theta,\ldots,t_s-\theta]]$, we have

$$L(\chi_{t_1}\cdots\chi_{t_s}) = \sum_{i_1,\ldots,i_s\geq 1} c_{i_1,\ldots,i_s}(t_1-\theta)^{i_1}\cdots(t_s-\theta)^{i_s}, \quad c_{i_1,\ldots,i_s} \in \mathbb{C}_\infty, \tag{36}$$

where on both sides, we have entire analytic functions (see Corollary 34). This can also be seen, alternatively, by considering the function F_{s-1} of Lemma 31, which vanishes, and observing that

$$L(\chi_{t_1}\cdots\chi_{t_s}, 1)|_{t_i=\theta} = F_{s-1}(t_1,\ldots,t_{i-1},t_{i+1},\ldots,t_s).$$

Let us focus on the coefficient $c_{1,\ldots,1}$ in the expansion (36). We then have

$$c_{1,\ldots,1} = \left(\frac{d}{dt_1}\cdots\frac{d}{dt_s}L(\chi_{t_1}\cdots\chi_{t_s})\right)\Big|_{t_1=\cdots=t_s=\theta}$$

so that

$$V_{1,s}(\theta,\ldots,\theta) = (-1)^s\widetilde{\pi}^{s-1}\sum_{d\geq 0}\sum_{a\in A^+(d)}\frac{a'^s}{a} = (-1)^s\widetilde{\pi}^{s-1}c_{1,\ldots,1} \in \mathbb{F}_q[\theta]$$

(by Corollary 34, the series on the right-hand side is convergent). Now, by Proposition 44, $\Pi(s)$ divides the polynomial $V_{1,s}(\theta,\ldots,\theta)$ in A. We then set, as in the introduction:

$$\mathbb{B}_s = \frac{V_{1,s}(\theta,\ldots,\theta)}{\Pi(s)} = G_s(\theta) \in A.$$

4.2 Proof of Theorem 5

We begin the proof with a couple of simple remarks. Firstly, if B is a polynomial of $A[t]$ and if $\mathfrak{p}$ is a prime of degree $d > 0$, then

$$\tau^d B \equiv B \pmod{\mathfrak{p}}.$$

The reason for this is that $\mathfrak{p}$ divides the polynomial $\theta^{q^d} - \theta$. In particular,

$$(\tau^d B)(\theta) \equiv B(\theta) \pmod{\mathfrak{p}}. \tag{37}$$

Secondly, recalling the $\mathbb{C}_\infty$-linear operator φ of subsection 4.1.1, we have

$$\tau\varphi = \varphi\tau = \rho,$$

where ρ is the operator defined by $\rho(x) = x^q$ for all $x \in \mathbb{C}_\infty((t))$. In particular, if $s = \sum_{i=0}^r s_i q^i$ is expanded in base q and if $d \geq r \geq i$, from

$$\tau^d \varphi^i = \tau^{d-i}\tau^i\varphi^i = \tau^{d-i}\rho^i$$

we deduce

$$(\tau^d \omega_s)(t) = \prod_{i=0}^r ((\tau^{d-i}\omega)(t))^{s_i q^i},$$

so that

$$(\tau^d \omega_s)(t) = \prod_{i=0}^r G_{d-i}(t,\theta)^{s_i q^i} \omega(t)^s. \tag{38}$$

We can finish the proof of Theorem 5. By (37),

$$\mathbb{B}_s \equiv (\tau^d W_s)(\theta).$$

We shall now compute $(\tau^d W_s)(\theta)$. If $d > r$, we can write

$$G_{d-i}(t,\theta)^{s_i q^i} = (t-\theta)^{s_i q^i} \prod_{j=1}^{d-i-1} (t - \theta^{q^j})^{s_i q^i},$$

and

$$\prod_{i=0}^r G_{d-i}(t,\theta)^{s_i q^i} = (t-\theta)^s F(t),$$

where $F(t)$ is a polynomial such that

$$F(\theta) = \prod_{i=0}^r l_{d-i-1}^{s_i q^i}.$$

Since

$$(\tau^d W_s)(t) = \widetilde{\pi}^{-q^d} L(\chi_t^s, q^d)(t-\theta)^s \omega(t)^s F(t)$$

and $\lim_{t\to\theta}(t-\theta)\omega(t) = -\widetilde{\pi}$, we get

$$\begin{aligned}\lim_{t\to\theta}(\tau^d W_s)(t) &= (-1)^s \widetilde{\pi}^{-q^d}\zeta(q^d-s)\widetilde{\pi}^s \prod_{i=0}^{r} l_{d-i-1}^{s_i q^i} \\ &= (-1)^s \frac{BC_{q^d-s}}{\Pi(q^d-s)} \prod_{i=0}^{r} l_{d-i-1}^{s_i q^i}.\end{aligned}$$

Our Theorem 5 follows at once. □

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